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MEDIUM-TERM MONEY NEUTRALITY AND THE EFFECTIVE LOWER BOUND

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ABSTRACT

Conventional wisdom suggests that medium-term money neutrality imposes strong limitations on the effects of monetary policy. The point of this paper is that models with medium- and long-term money neutrality are prone to generate non-existence of equilibria at the effective lower bound (ELB) on interest rates. Non-existence is suggestive of sharp output contractions --- so-called contractionary black holes --- at the ELB. Paradoxically, the case for expansionary monetary policy at the ELB is even stronger in models that feature near money neutrality. The results highlight the benefits of a monetary policy regime in which the central bank temporarily overshoots its inflation target once confronted by the ELB.

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1 Introduction

"What we know, or should know, from the past is that once inflation becomes anticipated and ingrained — as it eventually would — then the stimulating effects are lost." (Paul Volcker, former Chairman of the Federal Reserve, New York Times, 9/19/11)

Most modern macroeconomic models feature medium- or long-term monetary neutrality or are relatively close to neutrality. This means, one might suspect, that with sufficiently long time passing, output is pinned down by the production capacities and labor supply, independently of monetary policy. The monetary neutrality assumption made its way into the core of economic thought following the seminal contributions of Friedman (1968) and Phelps (1967) and the stagflation of the 1970s when inflation rose without any improvement in output and employment.

The point of this paper is that this medium- or long-term monetary neutrality assumption runs into a direct conflict with the aggregate demand side of modern general equilibrium models under relatively plausible conditions. The two most important conditions for this conflict to occur are that i) central bank reaches the effective lower bound (ELB) on interest rates, and ii) policy is set according to a monetary policy regime under which the central bank is strongly averse to any overshooting of its in inflation target. More specifically, we consider a policy regime according to which the central bank is unable/unwilling to increase inflation expectations in response to a shock that brings the economy to the ELB. A policy regime governed by the popular Taylor rule, for example, satisfies this condition.

The conflict between aggregate supply and demand results in some cases in non-existence of equilibria, or in output contracting without a bound at the ELB — a phenomenon we term contractionary black holes. As we argue below, the reason is that any adverse shock that brings the economy to the ELB and causes a decline in short-run inflation and medium-run inflation expectations induces an increase in the real interest rate, which in turn reinforces the output contraction and the downward pressures on inflation and inflation expectations. A key conclusion of this paper is that the closer the aggregate supply side is to exhibiting medium- or short-term money neutrality, the more likely the model is to exhibit a very large output contraction at the ELB, or non-existence of the equilibrium. In addition, the case for expansionary monetary policy at the ELB is even stronger, and the stimulative effects of monetary policy are also larger, the closer the model is to exhibit money neutrality. This may seem deeply counterintuitive.[1] At the ELB, Volcker's opening quote of the paper is turned on its head: as expectations become more ingrained the effect of a stimulus becomes larger, rather than smaller.

We start the paper by showing the problem of non-existence in a microfounded model which nests a standard Phillips curve with a static inflation output trade-off, and the New Classical Phillips curve where there is no such trade-off once expectations adjust:

$$\pi_t = \kappa \hat{Y}_t + \lambda E_{t-1}\pi_t, \tag{1}$$

[1] Aside from Volcker who is quoted at the beginning of this paper, many other prominent economists have expressed skepticism about policies intended to raise inflation expectations at the ELB. For instance Cochrane (2012): "But it's a rare Phillips curve in which raising expected inflation is a good thing. It just gives you more inflation, with if anything less output and employment."

where $\kappa > 0, \lambda \in [0,1]$ are coefficients, π_t is inflation in deviation from steady state, $\hat{Y}_t$ is output in deviation from its potential, and E_{t-1} is an expectations operator. Our benchmark is $\lambda = 1$, in which case the model is the standard New Classical Phillips curve with strong monetary neutrality (Lucas, 1972; Barro and Gordon, 1983). Meanwhile, if $\lambda = 0$ the model features a classic static trade-off between inflation and output. We choose this model, not for its realism, but for clarity. It clearly illustrates the problem of non-existence when the aggregate supply side incorporates a strong money neutrality. Here, if inflation is fully anticipated and $\lambda = 1$, then $\pi_t = E_{t-1}\pi_t$ and equation (1) implies that $\hat{Y}_t = 0$. Hence, any increase in inflation once anticipated, seems to loose its expansionary effect, as suggested by Volcker's quote.

We point out in Section 3 that this particular result is obtained by looking exclusively at the "supply equation" (1) under the assumption that $\lambda = 1$. When we introduce a standard aggregate demand side to the model, along with a standard monetary policy reaction function, it becomes clear that this result is only partial and misleading in general equilibrium. A key result is that if shocks are large enough for the ELB on the short-term nominal interest rate to bind, then, given expectations, output is demand determined. In that case — assuming monetary policy seeks to bring inflation back to the central bank's inflation target without overshooting it — output demanded is always below "potential," i.e., $\hat{Y}_t < 0$. Moreover, even if inflation is perfectly anticipated, we show that the solution $\hat{Y}_t = 0$ implied by the supply side is inconsistent with the demand-side equilibrium conditions of the model. Hence a key result of this paper is the non-existence of an equilibrium in the model when the ELB binds.

How can this clash between the demand and supply sides of the model be resolved? What is the interpretation of the non-existence result? We propose an interpretation by analyzing two cases that each relax an assumption of our baseline model. First, in Section 4, we relax the assumption that $\lambda = 1$. We show that the non-existence result is a knife-edge result for $\lambda = 1$, and an equilibrium exists in close vicinity of it, although it requires a sharp drop in output. Indeed, paradoxically, as the model moves closer to monetary neutrality, i.e., as λ moves from 0 to 1, the output in the short run contracts more and more until it collapses without bound. Second, while maintaining the benchmark assumption that $\lambda = 1$, we relax in Section 5 the assumption of perfect foresight. It turns out that non-existence in a perfect foresight equilibrium is also a fragile knife-edge result. We show that when introducing a little uncertainty, again an equilibrium exists. In this equilibrium the "clash of the two equations" is again resolved in favor of the demand side, and the solution features an output collapse and deflation. Once inflation becomes perfectly anticipated towards the perfect foresight limit, output and inflation collapse completely. We refer to the situation in which output collapses without a bound as a contractionary black hole.

Interestingly, we show that at the ELB, as long as the equilibrium exists, anticipated inflation is far from neutral under our specification of the monetary policy regime. In fact, raising inflation expectations can have extremely large effects on output in this model, with the effects getting larger and larger as the model approaches the contractionary black hole. This implies that the monetary policy regime plays a key role in stabilizing output and inflation. In our baseline, we assume that, when possible, the central bank sets its policy rate according to a simple policy rule that seeks to bring inflation back to its target (from below) in response to a ELB shock. However, if the policy regime instead implies a temporary overshoot of the inflation target it can successfully stabilize inflation and output. If properly designed, a new policy regime can even eliminate the possibility of contractionary black holes altogether.

Our paradoxical conclusion that output contracts more at the ELB the closer the model is to monetary neutrality is related to, but is conceptually distinct from the price flexibility paradox in New Keynesian models.[2] According to this paradox, if the central bank has an inflation target that is too low to accommodate a negative natural rate of interest, then the more prices become flexible, the more output contracts at the ELB. This is because higher price flexibility results in more deflationary expectations, which contract aggregate demand. This is paradoxical, for we know that at full flexibility, output is by definition equal to the flexible-price output. The resolution of this paradox, we suggest (and show explicitly toward the end of the paper), is that there is no equilibrium in the flexible-price model.

While most of our analysis is conducted in linear(ized) models, Section 6 shows that our main result on non-existence of the equilibrium is not an artifact of an approximation. Indeed, the equilibrium displays similar features in the non-linear version of our baseline model.

Section 7 moves to the more empirically realistic New Keynesian model now popular in the literature. In that model, the short run is no longer just defined by just one period. The way the non-existence arises is via a constraint on how long the slump can last without an unbounded collapse in output. For empirically reasonable values, for example, the bound on how long the slump can last without implosion is relatively tight. One implication of this result, from a modeling standpoint, is that one has to move away from assuming monetary neutrality in the medium and short run in order to account for plausibly persistent slumps. For example, it is difficult to account for the long slump in Japan at the ELB without deviating away from medium term monetary neutrality.

In Section 8, we connect our results to the literature on secular stagnation, a literature emphasizing a permanent liquidity trap. This section highlights that a key requirement of this literature is not only a change to the aggregate demand side of the model — in order to encompass a permanent reduction in the real interest rate — but also to the aggregate supply side, to allow for a permanent trade-off between inflation and output. In Section 9, we connect our result to the price-flexibility paradox. Here we highlight via a simple flexible-price model, that the key to understand the paradox is that there generally is non-existence of equilibria in response to temporary negative real interest rates if the central bank targets zero inflation. Section 10 discusses alternative policy rules that generate equilibrium existence, such as policies seeking to stabilize prices around a price-level target. Section 11 concludes.

2 Medium-Term Neutrality of Money

We start by briefly reviewing a key property of the aggregate supply relationship (1), shared by most modern specifications of nominal rigidities, namely the medium-run money neutrality when $\lambda = 1$. We label π_t^* the inflation rate that the central bank seeks to achieve in any period t, and call it the inflation target for short. We assume that we initially start with an inflation target at 0, and that the central bank raises the inflation target to $\pi_S^* > 0$ in the short run, that is at $t = S$. We also assume that in the short run, inflation expectations haven't adjusted, so that $E_{S-1}\pi_S = 0$. Let us define the medium run as the period in which expectations have adjusted so that $\pi_M = E_S\pi_M = \pi^*$. For now, there is no difference between long and medium run but we will make a distinction between these shortly.

Figure 1 shows the output and inflation rates in the model in the short run, given by the schedule

[2] This paradox was first shown in the New Keynesian model in Eggertsson (2012). See also discussion in Eggertsson and Krugman (2012), Werning (2012), Christiano, Eichenbaum and Rebelo (2011) and Bhattarai, Eggertsson Schoenle (2018). Mathematically, the flexibility paradox is about higher κ leading to lower output at the ELB, while this paper is about λ moving closer to 1, which leads to a fall in output.

$\pi_S = \pi^* = \kappa \hat{Y}_S$ so that $\pi^* = \kappa \hat{Y}_S$. We see that in the short run the government can achieve higher output by creating (unexpected) inflation, by increasing its inflation target π^*. The trade-off between inflation and output is given by κ^{-1}, i.e. a one percentage point increase in inflation increases output by κ^{-1} percent. Hence in the short run the government has a menu of choices of inflation output pairs on the solid line dotted by A and B, but this trade-off will generally depend on the extent to which the higher inflation targets is "ingrained" in expectations. For large enough inflation the government can even achieve the first best output $\hat{Y}^*$ that may be different from the steady state due, e.g., to monopoly distortions.

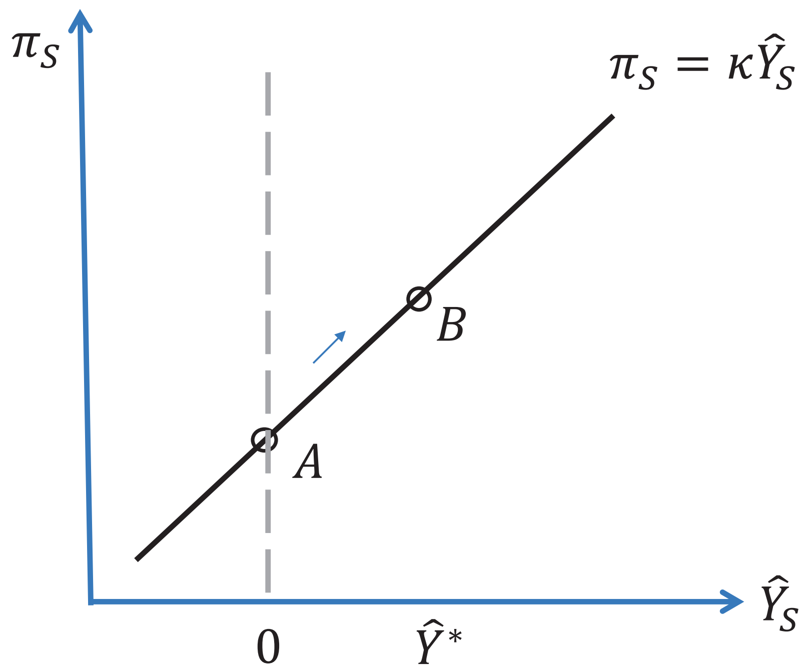

Figure 1: The tradeoff between inflation and output when inflation expectations are fixed.

The key lesson of models with forward-looking price setters is that this trade-off is an illusion once expectations adjust. Let us consider the baseline specification in which $\lambda = 1$. Once the new inflation target π^* becomes fully anticipated, as happens in the medium run, then $\pi_t = E_{t-1}\pi_t = \pi^*$ and the AS equation (1) becomes $\pi_M = \pi^* = \kappa \hat{Y}_M + \pi^*$, which implies that $\hat{Y}_M = 0$. Once an inflation policy is anticipated, the government can only choose between different inflation rates on dashed curve from A to C in Figure 2 without any improvement in output. In particular, suppose the private sector anticipated that the government would try to achieve point B (where output is at potential). In this case $E_S\pi_M = \pi_M$ and the AS curve shifts as shown in the figure so that once the government chooses π_M there are no gains in output, only higher inflation. This is at core the Volcker's remark "What we know, or should know, from the past is that once inflation becomes anticipated and ingrained — as it eventually would — then the stimulating effects are lost."

The same property holds in most modern models of dynamic price setting in absence of ad-hoc assumptions. We will review in detail in Section 7 the popular New Keynesian model. A leading alternative is the sticky information model of Mankiw and Reis (2002). According to their model, the aggregate supply

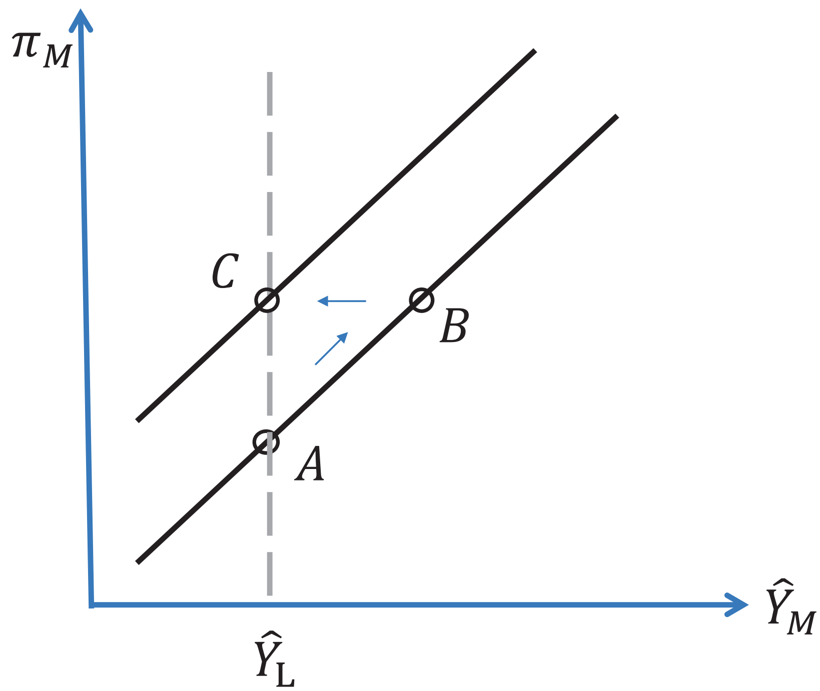

Figure 2: If inflation expectations adjust fully they eliminate any output gains from inflation.

curve takes the form

$$\pi_t = \frac{\eta\nu}{1-\nu}\hat{Y}_t + \nu\sum_{j=0}^{\infty}(1-\nu)^j E_{t-1-j}\left[\pi_t + \eta(\hat{Y}_t - \hat{Y}_{t-1})\right] \tag{2}$$

where $\eta > 0$ is a coefficient and $0 < \nu < 1$. This model can be seen as a generalization of the baseline specification just discussed (1): once again we see that once higher inflation is anticipated, the second term on the right-hand side of (2) reduces to π^*. In this model, however, it takes a longer time to obtain money neutrality than in the simple example that serves as our benchmark.

3 Introducing Demand: The Problem of Non-Existence

In the last section we reviewed the simple New Classical model that features no trade-off between inflation and output in the medium run (when $\lambda = 1$), where the medium run is defined by the fact that expectations have adjusted. Missing in this picture, however, is a demand side. Here we introduce the demand side, and formulate, in the process, microfoundations for the Phillips Curve.

3.1 Microfoundations

A representative household maximizes

$$E_t \sum_{T=t}^{\infty} \beta^{T-t}\left[u(C_T) - v(l_T)\right]\xi_T, \tag{3}$$

where $\beta \in (0,1)$ is a discount factor, $C_t \equiv \left[\int_0^1 c_t(j)^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}}$ a Dixit-Stiglitz aggregate with $\theta > 1$, $P_t \equiv \left[\int_0^1 p_t(j)^{1-\theta} dj\right]^{\frac{1}{1-\theta}}$ a price index, l_t labor and ξ_t a shock. $u(.)$ is an increasing concave function while $v(.)$ is increasing and convex. The period budget constraint is

$$P_t C_t + B_t = (1+i_{t-1})B_{t-1} + \int_0^1 Z_t(j)dj + P_t W_t l_t - T_t, \tag{4}$$

where B_t is a riskless bond, i_t the nominal interest rate, $Z_t(i)$ firms profits and W_t the wage. The household satisfies a borrowing limit to preclude Ponzi schemes:[3]

$$(1+i_t)B_t \geq -\sum_{T=t+1}^{\infty} E_{t+1}\left[Q_{t+1,T}\left(\int_0^1 Z_T(j)dj + P_T W_T l_T - T_T\right)\right] > -\infty, \tag{5}$$

where $Q_{t,T}$ denotes the stochastic discount factor.

The household's optimal plan satisfies

$$u_c(C_t)\xi_t = (1+i_t)\beta E_t\left[u_c(C_{t+1})\xi_{t+1}\Pi_{t+1}^{-1}\right] \tag{6}$$

where $\Pi_t \equiv P_t/P_{t-1}$ and

$$W_t = \frac{v_l(l_t)}{u_c(C_t)}, \tag{7}$$

and the nominal interest rate satisfies an effective lower bound[4]

$$i_t \geq 0. \tag{8}$$

There is a continuum of measure 1 of firms. A fraction γ_{fix} of firms set prices one period in advance, a fraction γ_{ind} sets prices equal to last period's aggregate price index P_{t-1}, and the remaining fraction $(1-\gamma_{fix}-\gamma_{ind})$ sets prices freely. A unit of labor produces one unit of output ($Y_t = l_t$). The preferences of households imply a demand for good i of the form $y_t(j) = Y_t(\frac{p_t(j)}{P_t})^{-\theta}$, where $Y_t = C_t$ is aggregate output. Firms maximize profits $Z_t(j) = p_t(j)Y_t(p_t(j)/P_t)^{-\theta} - W_t P_t Y_t(p_t(j)/P_t)^{-\theta}$, where j indexes the firm.

The prices chosen by the flexible price setters, the ones who sets prices one period in advance price, and those who index their price are respectively

$$\frac{p_t^{flex}}{P_t} = \frac{\theta}{\theta-1}W_t, \tag{9}$$

$$E_{t-1}\left[u_c\left(Y_t\right)Y_t\left(\frac{p_t^{fix}}{P_t}\right)^{-\theta}\left(\frac{p_t^{fix}}{P_t} - \frac{\theta}{\theta-1}W_t\right)\right] = 0, \tag{10}$$

$$p_t^{ind} = P_{t-1}. \tag{11}$$

The aggregate price index implies

$$1 = (1-\gamma_{fix}-\gamma_{ind})\left(\frac{p_t^{flex}}{P_t}\right)^{1-\theta} + \gamma_{fix}\left(\frac{p_t^{fix}}{P_t}\right)^{1-\theta} + \gamma_{ind}\left(\frac{P_{t-1}}{P_t}\right)^{1-\theta}. \tag{12}$$

An equilibrium is a set of stochastic processes $\{\frac{p_t^{flex}}{P_t}, \frac{p_t^{fix}}{P_t}, \Pi_t, Y_t, W_t, i_t\}$ that satisfy equations (6)–(10), (12), and the equilibrium condition $Y_t = l_t = C_t$, for a given shock process $\{\xi_t\}$ and for given fiscal and monetary policies that satisfy (5).

[3]See Woodford (2003, Chap 2.) for discussion.

[4]This constraint can be interpreted as a consequence of the household maximization problem if there exists money in the economy as a nominal store of value.

3.2 Approximated model

A log-linear approximation of optimal pricing conditions (9)-(12) yields the Phillips Curve (1) repeated here

$$\pi_t = \kappa \hat{Y}_t + \lambda E_{t-1}\pi_t, \tag{13}$$

where $\kappa > 0$ and $0 \leq \lambda \leq 1$, as shown in Appendix A.[5] The benchmark New Classical Phillips Curve obtains when $\lambda = 1$, that is if $\gamma_{ind} = 0$. The traditional Phillips Curve with static trade-offs ($\lambda = 0$) obtains if $\gamma_{fix} = 0$ and $\gamma_{ind} > 0$.

A log-linear approximation of the household's consumption Euler equation (6) yields the so-called IS equation

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(i_t - E_t\pi_{t+1} - r_t^e), \tag{14}$$

where $\hat{Y}_t \equiv \log\left(Y_t/\bar{Y}\right)$, $\pi_t \equiv \log\left(P_t/P_{t-1}\right)$, i_t refers to $\log(1+i_t)$, and $\sigma > 0$. The "natural" rate of interest $r_t^e \equiv \bar{r} + E_t \log\left(\xi_t/\xi_{t+1}\right)$ is an exogenous variable that depends on $\bar{r} \equiv \log\beta^{-1} > 0$ and ξ_t. The interest rate bound can once again be expressed as

$$i_t \geq 0.$$

Reviewing the aggregate supply relationship characterized above, it should be clear why previous authors have often abstracted from the demand side. To the extent that the nominal interest rate does not appear in equation (1), or in the government's assumed objective function, there is no loss of generality in assuming that instead of choosing the nominal interest rate, i_t, the government chooses directly either π_t or $\hat{Y}_t$ that satisfy the restriction (1). The nominal interest rate consistent with these levels of inflation and output can then just be backed out of (14).[6] However, nothing in this way of proceeding guarantees that the implied interest rate must be non-negative. Therefore, when one explicitly accounts for the ELB, aggregate demand needs to be incorporated.

Introducing aggregate demand requires a more complete specification of monetary policy. We assume that policy satisfies:

$$i_t = \max\{0, r_t^e + \pi_t^* + \phi_\pi(\pi_t - \pi_t^*)\} \tag{15}$$

where π_t^* is the government's inflation target, and $\phi_\pi > 1$ so that the so-called "Taylor principle" applies, whereby the nominal interest rate is raised more than one-for-one with inflation around the inflation target.

There are several reasons to focus on policy (15). One is that it implements the optimal monetary policy under discretion if the central bank minimizes[7]

$$L_t = E_t \sum_{T=t}^{\infty} \beta^{T-t}\{(\pi_T - \pi_T^*)^2 + \omega_y \hat{Y}_T^2\}.$$

A second is that this policy specification implies that at the ELB, the central bank does not try to commit to generating above-target inflation. Arguably, this is a reasonable characterization of the policy of major

[5] The coefficients are $\kappa \equiv \frac{(1-\gamma_{fix}-\gamma_{ind})\frac{1-\gamma_{ind}}{\gamma_{fix}}}{\left(1+(1-\gamma_{fix}-\gamma_{ind})\frac{\gamma_{ind}}{\gamma_{fix}}\right)}\left(\omega+\sigma^{-1}\right) > 0$, $\lambda \equiv \left(1+(1-\gamma_{fix}-\gamma_{ind})\frac{\gamma_{ind}}{\gamma_{fix}}\right)^{-1}$, where $\omega \equiv \frac{v_{ll}l}{v_l} > 0$ and $\sigma^{-1} \equiv -\frac{u_{cc}C}{u_c} > 0$.

[6] A similar argument can also be made for the money supply. We can add money in the utility function into our framework, so that the government's choice of the nominal interest rate is then modeled via its choice for the money supply. We omit this detail here.

[7] See Eggertsson (2006).

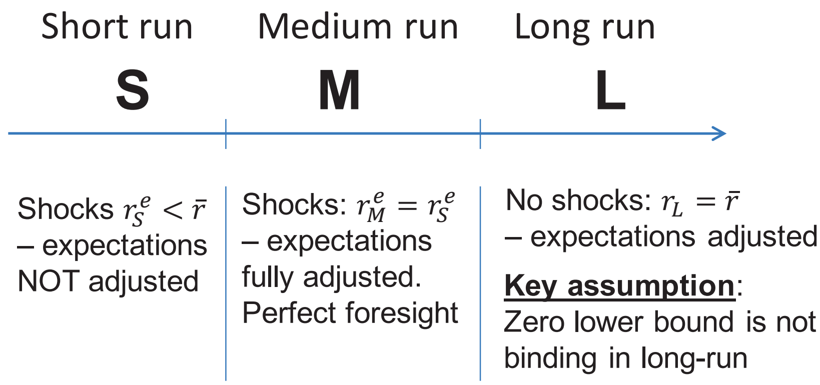

Figure 3: Time protocol according to Assumption 1.

central banks following the crisis of 2008, as no major central bank committed explicitly to generating above-target inflation, even if this would be optimal in the model, under commitment.[8] In Section 10, we discuss how the existence results reported below are affected by alternative policy rules.

Let us revisit the example discussed in Section 2, where we explored the effect of increasing the inflation target. How does aggregate demand complement the picture? As inflation becomes anticipated, the equilibrium moves from point B to C: Higher inflation leads to no output gains once fully anticipated. Consistently with equation (14), the increase in inflation is reflected in a higher nominal interest rate, i.e., $i_S = \pi_M$. This is arguably — in broad terms — what happened during the "great inflation" of the 1970s: as inflation expectations rose, nominal rates rose as well, with no gains in output. It would seem, then, at least when studying the 1970s, that the demand side is only relevant to "back out" the nominal interest rate.

However, when the ELB becomes binding, the demand side no longer just backs out the interest rate consistent with that equilibrium. Instead, with the interest rate fixed, equation (14) plays a key role in determining the *overall number of good demanded*. This is the case that we now turn to.

3.3 Short, medium, long run, and equilibrium non-existence in the New Classical benchmark

We proceed with the New Classical benchmark ($\lambda = 1$) to show how strong medium-term money neutrality generates equilibrium non-existence, in the face of the ELB. Consider an unexpected negative shock $r_t^e < \bar{r}$ in period zero called the "short run." The short run is defined by the fact that expectations have not adjusted; they remain at "steady state" so that $E_{-1}\pi_S = \pi_L^*$. The shock stays at its negative level in the next period (called the medium run) and reverts back to normal in the third period called the "long run." Expectations fully adjust in the medium and the long run so that the only difference between the medium and the long run is the absence of the shock in the long run. The only difference between the medium and the short run is that expectations have adjusted in the medium run, not in the short run. To summarize:

[8] One interpretation of that behavior is that central banks did not commit to generate above-target inflation because a commitment of that kind is not dynamically consistent, i.e., it is not "credible" if the public believes that the central bank sets policy under discretion.

A1 Consider the three periods $t = S, M, L$. In period $t = S$ there is an *unexpected* shock $r_t^e = r_S^e < \bar{r}$. In period $t = M$ the shock is still $r_t^e = r_S^e$. In periods $t \geq L$ the shock is back at steady state $r_t^e = r_L = \bar{r}$. While the shock is unexpected in period $t = S$, so that $E_{S-1}\pi_S = \pi^*$, there is perfect foresight between S, M, and L. Periods $t > L$ are identical to L: there are no shocks and agents have perfect foresight.

We proceed by analyzing the long run, before showing non-existence of equilibria in the medium run. The proposition below establishes that in the long run, $t \geq L$, policy rule (15) implies a unique bounded solution at positive interest rates in which $\pi_t = \pi_L^* = \pi^*$. We assume that ELB is not binding in the long run, thus excluding the possibility of self-fulfilling liquidity traps, a subject of another branch of the literature.[9] This restriction is relaxed in Section 6 below.

Proposition 1 *Suppose A1, that $\bar{r} > -\pi_L^*$, $\phi_\pi > 1$ and that the nominal interest rate is always positive in the long run $t \geq L$. Then the model (1), (14)–(15) implies a unique bounded long-run equilibrium $\{\pi_L, \hat{Y}_L, i_L, E_L\pi_{L+1}, E_L\hat{Y}_{L+1}\}$ given by $\pi_L = E_L\pi_{L+1} = \pi_L^*$, $i_L = \bar{r} + \pi_L^*$ and $\hat{Y}_L = E_L\hat{Y}_{L+1} = 0$.*

Proof. Given Assumption A1, perfect foresight between periods M and L implies that $E_{L-1}\pi_L = E_M\pi_L = \pi_L$. It follows from equation (1) that for all $t \geq L$

$$\pi_t = \kappa\hat{Y}_t + \pi_t$$

or simply that that $\hat{Y}_t = 0$ for $t \geq L$. This implies that for any $t \geq L$ (14) simplifies to

$$i_t = E_t\pi_{t+1} + \bar{r}. \tag{16}$$

It follows from (15) and (16) that

$$E_t\pi_{t+1} + \bar{r} = \bar{r} + \pi_L^* + \phi_\pi(\pi_t - \pi_L^*)$$

or equivalently that

$$\pi_t = \phi_\pi^{-1}E_t\pi_{t+1} + (1 - \phi_\pi^{-1})\pi_L^*.$$

Iterating this forward yields

$$\pi_t = \phi_\pi^{-j}E_t\pi_{t+j} + (1 - \phi_\pi^{-1})\pi_L^* \sum_{s=1}^{j} \phi_\pi^{-(s-1)} = \pi_L^*$$

for any bounded $\{\pi_t\}_{t \geq L}$ since $\phi_\pi > 1$ and $\lim_{j\to\infty} \phi_\pi^{-j}E_t\pi_{t+j} = 0$. ■

Consider the medium run. The aggregate supply (AS) equation (1) implies

$$\hat{Y}_M = \kappa^{-1}(\pi_M - E_S\pi_M) = \kappa^{-1}(\pi_M - \pi_M) = 0$$

where the last equation follows from perfect foresight between the short and medium run in A1.

The aggregate demand (AD) equation is obtained by combining (15) with (14) to yield

$$\hat{Y}_M = \begin{cases} -\sigma\phi_\pi(\pi_M - \pi_M^*) + \sigma(\pi_L^* - \pi_M^*) & \text{if } r_M^e + \pi_M^* + \phi_\pi(\pi_M - \pi_M^*) > 0 \\ \sigma r_M^e + \sigma\pi_L^* & \text{if } r_M^e + \pi_M^* + \phi_\pi(\pi_M - \pi_M^*) \leq 0 \end{cases}. \tag{17}$$

[9] Eggertsson and Woodford (2003), for instance, suggest some policies which can exclude these type of equilibria.

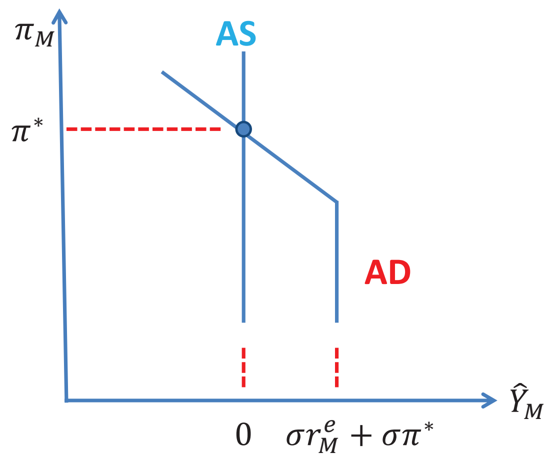

Figure 4: Equilibrium at positive interest rate.

The two curves are plotted in Figure 4 for a realization of the shock r_M^e satisfying $r_M^e + \pi_M^* + \phi_\pi(\pi_M - \pi_M^*) > 0$. Since inflation is perfectly anticipated the AS curve is vertical. Meanwhile, the AD curve is downward sloping in inflation. This is due to the fact that $\phi_\pi > 1$ so that the central bank reduces the policy rate more than one-to-one with the fall in inflation (and vice versa), thus stimulating spending as inflation drops, as shown in the first row of (17). Yet, there is a limit to how much the central bank can stimulate spending by rate cuts. If medium term inflation, π_M, is low enough so that the ELB is binding, then $\hat{Y}_M$ is given by the second row of (17), i.e. $\hat{Y}_M = \sigma r_M^e + \sigma\pi_L^*$. Now output demanded does not depend upon realized inflation (the ratio of prices today relative to yesterday). Instead demand only depends upon *expected inflation*, $E_M\pi_L\,(= \pi_L^*)$, which determines the price of goods tomorrow relative to the price today.

An equilibrium is determined by the intersection of the AS and AD equation. For a positive interest rate, we see that this equilibrium determination happens at $\pi_M = \pi_M^*$ and $\hat{Y}_M = 0$. Figure 5 shows the effect of the shock r_M^e being more negative. This shifts the AD curve leftward. If this shock is small enough, the nominal interest rate is reduced but inflation stays at π_M^* and output at potential.

There is nothing in the model, however, that prevents the AD curve from shifting even further than in Figure 5. Consider $r_M^e < -\pi_M^*$. This shifts the AD curve to the left of the AS curve, a situation shown in 6. Clearly the two curves do not intersect. In other words *there is no equilibrium.* To summarize:

Proposition 2 *Suppose A1 and* $\lambda = 1$, $r_M^e < -\pi_L^*$, $\bar{r} > -\pi_L^*$, $\phi_\pi > 1$, *and that the nominal interest rate is always positive in the long run. Then there exists no bounded equilibrium in the medium run that satisfies equations (1) and (14)–(15).*

Proof. See Appendix. ■

How should this proposition be interpreted? The aggregate supply equation and the aggregate demand equation in Figure 5 are pointing to different directions. On the one hand, the aggregate supply equation

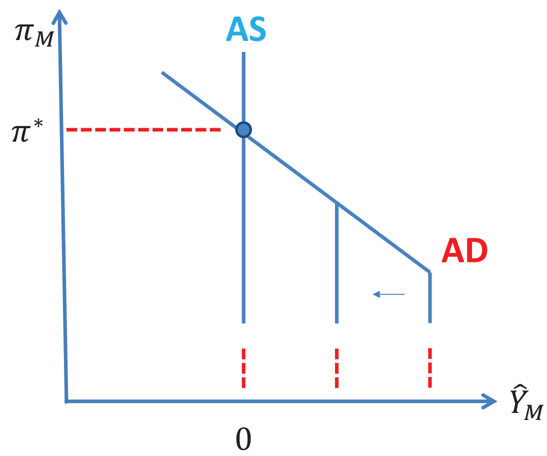

Figure 5: A shock to r_S^e has no effect on output or inflation as long as the zero bound is not binding.

imposes that $\hat{Y}_M = 0$: If everybody perfectly anticipates the future, then prices act *as if* they are perfectly flexible and output is thus at potential. Meanwhile, the aggregate demand equation imposes that demand must be below its steady state, i.e., $\hat{Y}_M < 0$. For households to be willing to buy all supplied goods, the real interest rate must be sufficiently negative. Yet, this is not possible if expected inflation is below π_t^* without a negative nominal interest rate. Evidently demand and supply clash — no level of output and inflation satisfy both equations at the same time. What is particularly curious is that some of the policy discussion reviewed in the introduction seems to be driven by an intuition which is derived exactly when this clash occurs but only using the AS equation. The interpretation we propose is that the non-existence result is suggestive of a severe output contraction that becomes visible once the equilibrium existence is restored. We do so in the next two sections, by relaxing the strong assumption of either i) full monetary neutrality as in the New Classical benchmark ($\lambda = 1$), or ii) perfect foresight.

Before proceeding further, it is worth pointing out another fundamental driving force of the non-existence result, namely monetary policy. To see this, note the requirement for non-existence of the equilibrium: $r_M^e < -\pi_L^*$. This means that the government can always guarantee the existence of an equilibrium or mitigate a very severe output contraction caused by the ELB, by choosing a high enough inflation target. While a higher long-run inflation target is one way of getting there, we also discuss more broadly other types of monetary policies in Section 10.

4 Restoring the Equilibrium via Monetary Non-Neutrality

The simplest way to restore an equilibrium is by departing from our New Classical benchmark, that is, by assuming that $0 \leq \lambda < 1$, so that (1) implies a trade-off between inflation and output in the medium run

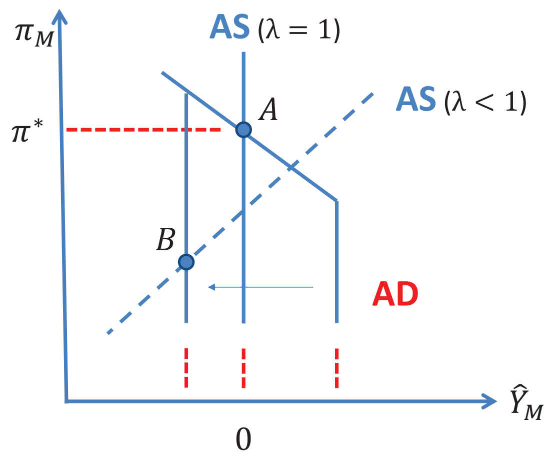

Figure 6: Aggregate demand and aggregate supply clash: No equilibrium.

given by

$$\pi_M = \frac{\kappa}{1-\lambda}\hat{Y}_M.$$

That trade-off restores equilibrium existence as can be inferred from Figure 6 where the dashed AS curve denotes the case when $\lambda < 1$: Once the AS curve is upward sloping it must intersect the aggregate demand curve. The equilibrium is one in which medium-term output is completely determined by the AD equation (17)

$$\hat{Y}_M = \sigma r_M^e + \sigma \pi_L^*, \tag{18}$$

while the AS equation pins down inflation

$$\pi_M = \frac{\kappa}{1-\lambda}\hat{Y}_M = \frac{\kappa}{1-\lambda}\sigma r_M^e + \frac{\kappa}{1-\lambda}\sigma \pi_L^*. \tag{19}$$

Equation (19) reveals that as the model moves closer to monetary neutrality, then medium-term inflation drops more and more, so that $\pi_M \to -\infty$ as $\lambda \to 1$.

Taking the medium-run solution for π_M and $\hat{Y}_M$ as given, output and inflation in the short run must satisfy the IS and AS equations

$$\begin{aligned} \hat{Y}_S &= \hat{Y}_M + \sigma \pi_M + \sigma r_S^e \\ \pi_S &= \kappa \hat{Y}_S. \end{aligned}$$

While a low natural rate of interest r_S^e contributes to an output contraction, the latter is reinforced by declining inflation in the medium run. In particular, as the Phillips Curve approaches the New Classical benchmark, the output contraction in the short run can be very large, given the large contraction in medium-run inflation. Substituting (18) and (19) into the IS curve in the short run, we obtain

$$\hat{Y}_S = \hat{Y}_M + \sigma \pi_M + \sigma r_S^e = \sigma\left(2 + \frac{\sigma\kappa}{1-\lambda}\right) r_S^e + \sigma\left(1 + \frac{\sigma\kappa}{1-\lambda}\right)\pi_L^*. \tag{20}$$

This expression leads directly to a central proposition of the paper.

Proposition 3 *Assume A1, $r_S^e + \pi_L^* < 0$, $\phi_\pi > 1$, and $\lambda < 1$. The higher λ, the lower output in the short run. As the model approaches the New Classical benchmark that features no medium-term monetary neutrality ($\lambda \to 1$), then output and inflation in the short run contracts without a bound, $\hat{Y}_S \to -\infty$ and $\pi_S \to -\infty$.*

This suggests that, as mentioned in the introduction, the equilibrium non-existence obtained as $\lambda \to 1$ is associated with a collapse in output and inflation, a so-called contractionary black hole. Thus as the model converges towards monetary neutrality, the fall in output intensifies without a bound.

A key policy implication of the above analysis is that the benefits of anticipated inflation become arbitrarily large, when the model approaches money neutrality. The following proposition follows directly from (20).

Proposition 4 *Suppose A1, $r_S^e < -\pi_M^*$, $\bar{r} > -\pi_L^*$, $\phi_\pi > 1$, $\lambda \leq 1$, and that the nominal interest rate is always positive in the long run. Then the ELB is binding and the effect of anticipated long-run inflation on output in the short run, $\frac{d\hat{Y}_S}{d\pi_L}$, is $\frac{\sigma^2\kappa}{1-\lambda}$. The output effect of anticipated inflation increases without a bound as $\lambda \to 1$.*

This proposition represents another fundamental result. As the model moves closer to the New Classical benchmark featuring medium-run monetary neutrality, then the benefits of increasing anticipated inflation increases without a bound as $\lambda \to 1$, when the interest rate is constrained by the ELB.

5 Restoring the Equilibrium via Uncertainty

Our analysis has so far abstracted from uncertainty. As shown in Section 3, the New Classical benchmark ($\lambda = 1$) features money neutrality in the medium run. However, monetary neutrality relies on the fact that people can perfectly forecast inflation. Realistically, however, there is always some uncertainty. The presence of uncertainty also implies a trade-off between output and inflation in the medium run, just as in the case that $\lambda < 1$. To illustrate this, we return to our New Classical benchmark with $\lambda = 1$, but deviate from perfect foresight, while maintaining rational expectations.

Instead of the shock staying "on" in the medium run with certainty before reverting back to steady state there is a probability α that the shock reverts back to steady state in the medium run. The structure of uncertainty is shown in Figure 7. As we now show, this seemingly minor extension generates existence.

A2 Consider three periods $t = S, M, L$. In period $t = S$ there is an *unexpected* shock $r_t^e = r_S^e < \bar{r}$. In period $t = M$ the shock is still $r_t^e = r_S^e$ with probability $(1-\alpha)$ and $r_t^e = r_L^e > 0$ with probability α. In periods $t \geq L$ the shock is back at steady state $r_t^e = r_L^e = \bar{r}$. The shock is unexpected in period $t = S$ but people form rational expectations about the shock in period $t = M$ using the correct probability distribution of the model (α).

The long run in the New Classical model ($\lambda = 1$) is as before $\hat{Y}_L = 0$, $\pi_L = \pi_L^*$ and $i_L = r_L^e + \pi_L^*$. In the medium run, there are two possible states, i) that the shock reverts back to steady state $r_M^e = r_L^e > 0$

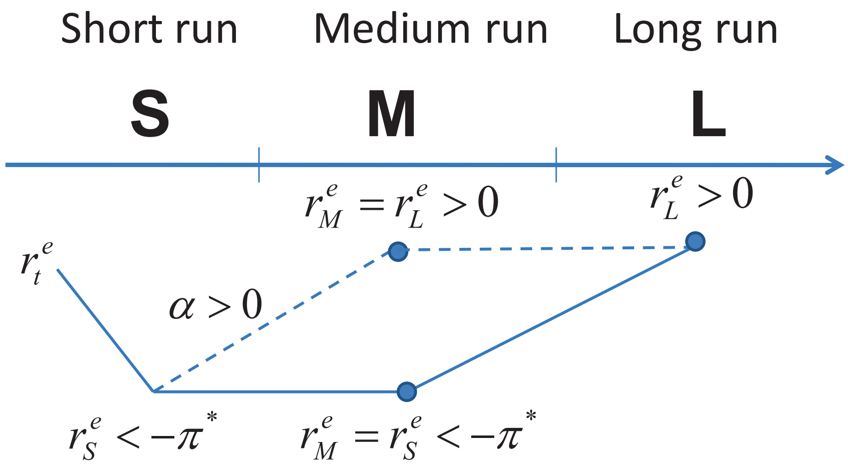

Figure 7: Introducing uncertainty. With probability α, the natural rate r_M^e in the medium run reaches its long run value $r_L^e > 0$. With probability $1-\alpha$ it remains at $r_S^e < -\pi^*$.

(which we call "high") or ii) that the shock remains at $r_M^e = r_S^e < -\pi_M^*$, in which case the nominal interest rate is constrained by the ELB. The model then solves the following six equations in the medium run

$$\pi_M^j = \kappa \hat{Y}_M^j + \alpha \pi_M^{high} + (1-\alpha)\pi_M^{low}, \quad \text{for } j = \textit{ low or high} \tag{21}$$

$$\hat{Y}_M^{low} = \hat{Y}_L - \sigma(i_M^{low} - \pi_L^* - r_S^e) \tag{22}$$

$$\hat{Y}_M^{high} = \hat{Y}_L - \sigma(i_M^{high} - \pi_L^* - r_L^e) \tag{23}$$

$$i_M^{low} = 0 \quad \text{and} \quad i_M^{high} = r_L^e + \pi_M^* + \phi_\pi(\pi_M^{high} - \pi_M^*), \tag{24}$$

while in the short run it solves

$$\hat{Y}_S = \alpha \hat{Y}_M^{high} + (1-\alpha)\hat{Y}_M^{low} + \sigma\alpha\pi_M^{high} + \sigma(1-\alpha)\pi_M^{low} + \sigma r_S^e \tag{25}$$

$$\pi_S = \kappa \hat{Y}_S. \tag{26}$$

The model's equilibrium is characterized in the following proposition.

Proposition 5 *Suppose A2, that $r_S^e < -\pi_M^*$, $\bar{r} > -\pi_L^*$, $\phi_\pi > 1$ and that the nominal interest rate is always positive in the long run $t \geq L$. Then with $\alpha > 0$ there exists a unique bounded solution to (21)–(26) given by*

$$\begin{aligned}
\hat{Y}_M^{high} &= -\frac{1-\alpha}{\alpha}\sigma\left(r_S^e + \pi_L^*\right) \\
\pi_M^{high} &= \frac{1-\alpha}{\phi_\pi \alpha} r_S^e + \frac{1}{\alpha\phi_\pi}\pi_L^* + \frac{\phi_\pi - 1}{\phi_\pi}\pi_M^* \\
\hat{Y}_M^{low} &= \sigma\left(r_S^e + \pi_L^*\right) \\
\pi_M^{low} &= \frac{1-\alpha+\phi_\pi\kappa\sigma}{\phi_\pi\alpha} r_S^e + \frac{1+\phi_\pi\kappa\sigma}{\phi_\pi\alpha}\pi_L^* + \frac{\phi_\pi - 1}{\phi_\pi}\pi_M^* \\
\hat{Y}_S &= \sigma\left(\frac{1-\alpha}{\alpha}\frac{1+\phi_\pi\kappa\sigma}{\phi_\pi} + 1\right) r_S^e + \sigma\frac{(1-\alpha)\phi_\pi\kappa\sigma + 1}{\phi_\pi\alpha}\pi_L^* + \sigma\frac{\phi_\pi - 1}{\phi_\pi}\pi_M^* \\
\pi_S &= \kappa\hat{Y}_S.
\end{aligned}$$

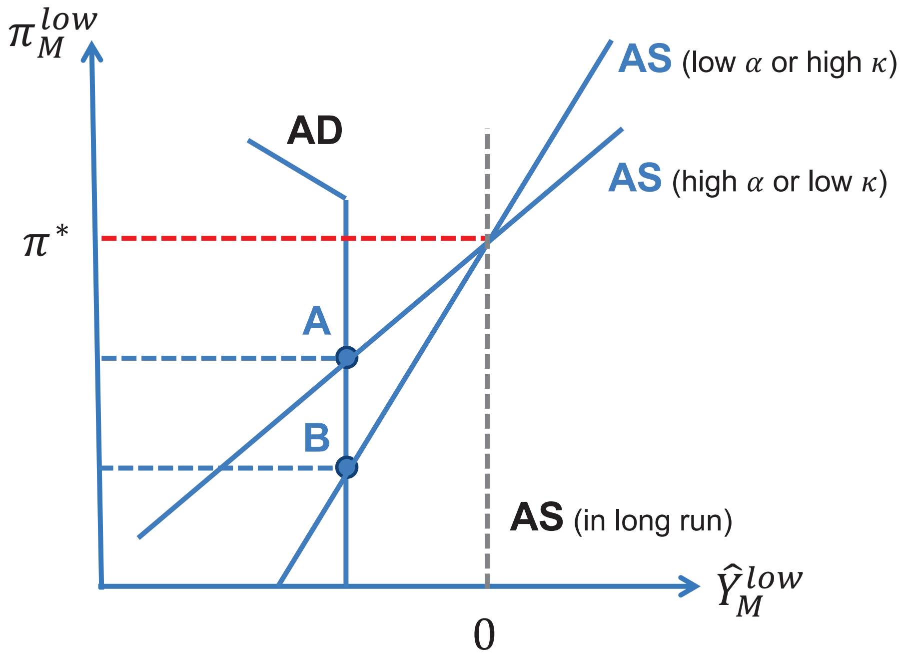

Figure 8: The clash between aggregate demand and aggregate supply resolved: Demand wins.

Proof. The solution is obtained by solving (21)–(26) (see Appendix for details). ■

The model equilibrium exists in the medium run, now that there is no longer perfect foresight. For instance, if the shock is in the low state, then the AS equation is given by

$$\pi_M^{low} = \kappa \hat{Y}_M^{low} + \alpha \pi_M^{high} + (1-\alpha)\pi_M^{low}$$

or

$$\pi_M^{low} = \frac{\kappa}{\alpha} \hat{Y}_M^{low} + \pi_M^{high}.$$

We can now use our solution for π_M^{high} to show that inflation and output are related in the medium run according to

$$\pi_M^{low} = \frac{1-\alpha+\phi_\pi \kappa\sigma}{\phi_\pi \alpha} \sigma^{-1} \hat{Y}_M^{low} + \phi_\pi^{-1}\pi_L^* + \left(1-\phi_\pi^{-1}\right)\pi_M^*. \tag{27}$$

Equation (27) reveals that the AS curve is no longer vertical, but that it is upward sloping in the output-inflation space as shown in Figure 8. What generates existence is the fact that inflation is no longer perfectly anticipated, as there is more than one state of the world in the medium run. Thus the presence of uncertainty generates equilibrium existence similarly to hard wiring a permanent trade-off between inflation and output, as we did in last section. Observe that as the uncertainty shrinks near the low state, then the amount of deflation in the medium term increases. Indeed, as the model approaches perfect foresight, i.e., $\alpha \to 0$ then $\pi_M \to -\infty$, just as in the case when $\lambda \to 1$.

Again, we will see that the limiting case in which uncertainty disappears features money neutrality in the medium run and very large contractions in economic activity and inflation, in the short run, when the interest rate is constrained by the ELB. Taking the medium-run solution for π_M and $\hat{Y}_M$ as given, output

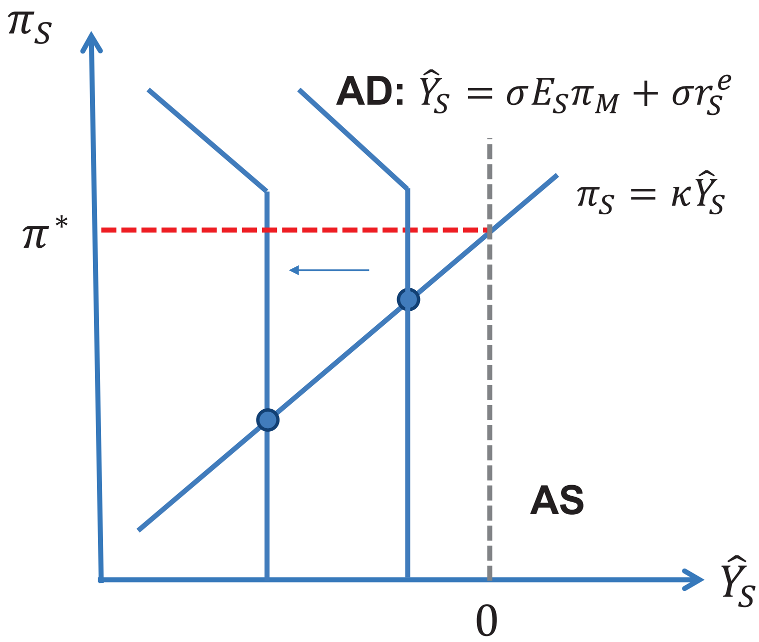

Figure 9: Deflation in the medium run reduces short run output since expected deflation increases the real interest rate and thus contracts demand.

and inflation in the short-run must satisfy the IS and AS equations

$$\begin{aligned}\hat{Y}_S &= E_S\hat{Y}_M + \sigma E_S\pi_M + \sigma r_S^e \\ \pi_S &= \kappa\hat{Y}_S,\end{aligned}$$

which are are plotted in Figure 9.

This figure looks similar to Figure 8. There is an important new element, however, lurking in the background. The AD curve shifts back not only due to the shock r_S^e, but also due to expected deflation — given by $E_S\pi_M$. The expected deflation in the model with uncertainty is given by

$$E_S\pi_M = \frac{(1-\alpha)\,(\kappa\sigma\phi_\pi + 1)}{\alpha\phi_\pi} r_S^e + \frac{\phi_\pi - 1}{\phi_\pi}\pi_M^* + \frac{(1-\alpha)\kappa\sigma\phi_\pi + 1}{\alpha\phi_\pi}\pi_L^*. \tag{28}$$

Again, medium term inflation expectation are declining as the model approaches monetary neutrality, which occurs here as it approaches perfect foresight $\alpha \to 0$ — ultimately dropping without a bound, as can be seen in expression (28). The next proposition summarizes this result for the New Classical benchmark ($\lambda = 1$), once we deviate from perfect foresight.

Proposition 6 *Assume A2, $r_S^e + \pi_L^* < 0$, $\phi_\pi > 1$, and $\lambda = 1$. As the probability of the medium-run recession increases, i.e., $\alpha \to 0$, then the output in the short run contracts without a bound $\hat{Y}_S \to -\infty$.*

Proof. The expression for $\hat{Y}_S$ in proposition 5 can be rewritten as $\hat{Y}_S = \sigma\left(\frac{(1-\alpha)\phi_\pi\kappa\sigma+1}{\phi_\pi\alpha}\right)(r_S^e + \pi_L^*) + \sigma\left(1-\phi_\pi^{-1}\right)(r_S^e + \pi_M^*)$. Since $r_S^e + \pi_L^* < 0$ by assumption, $\lim_{\alpha\to 0}\hat{Y}_S = -\infty$. ■

The last proposition is illustrated in Figure 10. We see that as the probability α converges to zero, output collapses. Meanwhile as this probability approaches 1, output converges to $\sigma(r_S^e + \phi_\pi^{-1}\pi_L^* + \left(1-\phi_\pi^{-1}\right)\pi_M^*)$. We could equivalently draw a similar figure for $\lambda \to 1$.

Note the parallel between the two departures from money neutrality considered in this and the previous section. Whether we let the model approach full neutrality via the fraction of price-setters who index their prices to lagged prices ($\lambda \to 1$) or by converging to the perfect foresight case ($\alpha \to 0$), output and inflation in the short run collapse without bound when the interest rate is constrained by the ELB.

Similarly to the previous section, the benefits of anticipated inflation become arbitrarily large, when the model approaches money neutrality:

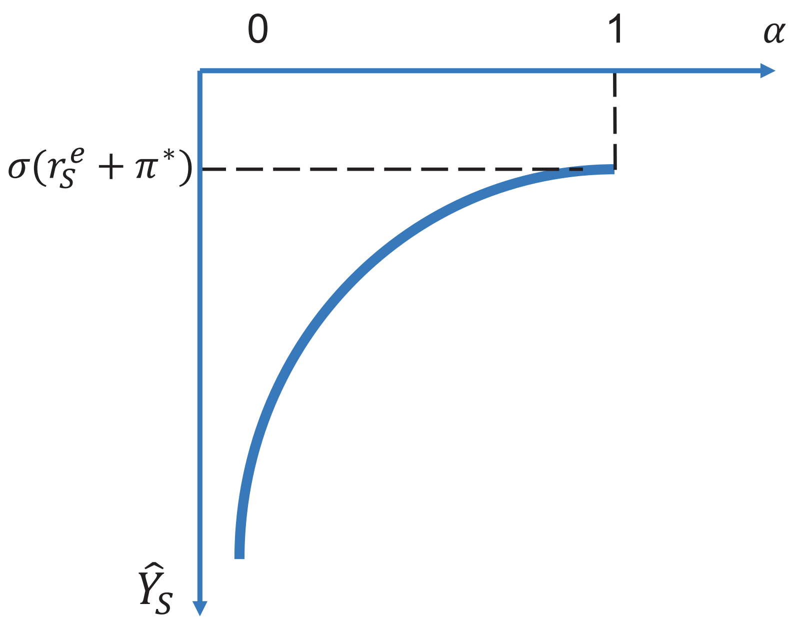

Figure 10: Output in the short run collapses as $\alpha \to 0$.

Proposition 7 *Suppose A2, $r_S^e < -\pi_M^*$, $\bar{r} > -\pi_L^*$, $\phi_\pi > 1$, $\lambda = 1$, and that the nominal interest rate is always positive in the long run. Then the ELB is binding and the effect of anticipated long-run inflation on output in the short run, $\frac{d\hat{Y}_S}{dE_S\pi_L}$, is $\sigma\frac{(1-\alpha)\kappa\sigma\phi_\pi+1}{\alpha\phi_\pi}$. This effect becomes arbitrarily large as $\alpha \to 0$.*

Proof. See Appendix. ■

As the New Classical model converges to perfect foresight, i.e., expected inflation becomes ”ingrained”, then the benefits of increasing long-run inflation expectations on short-term output become arbitrarily large.

6 Equilibrium Non-Existence in a Non-Linear Version of the Model

While our analysis has been conducted in the context of a linearized model the results are not an artifact of an approximation. To illustrate this we briefly mention a few results, and in particular the equilibrium non-existence when the ELB binds, in the New Classical benchmark, that is when $\gamma_{ind} = 0$ in the microfounded model (corresponding to $\lambda = 1$).

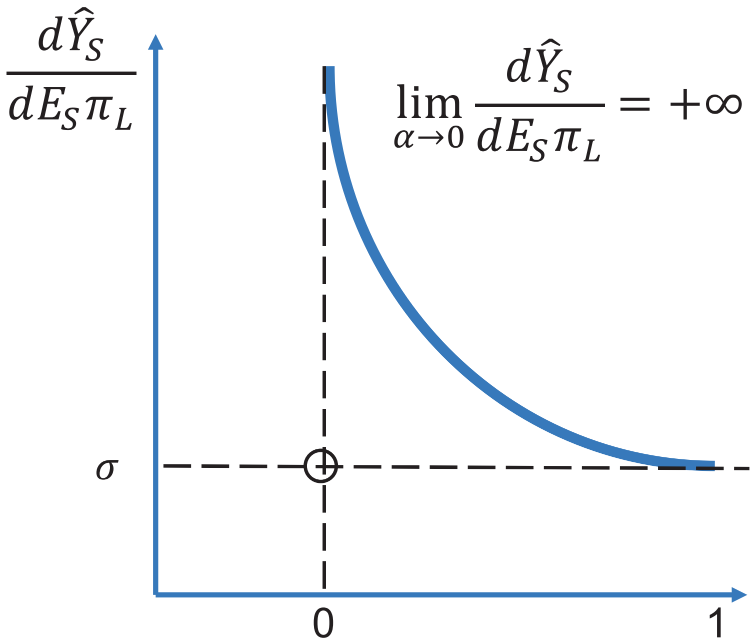

Figure 11: The output effect of inflation goes to infinity as $\alpha \to 0$.

The following proposition, which relates to Benhabib, Schmitt-Grohé and Uribe (2001), establishes the existence of multiple long-run steady states.

Proposition 8 *Consider the model given by (5)–(12) for all $t \geq t_0$. Suppose that monetary policy is given by an interest-rate reaction function*

$$i_t = \phi\left(\Pi_t, \Pi_t^*, \xi_t\right) \tag{29}$$

where $\Pi_t \equiv P_t/P_{t-1}$ denotes inflation, Π_t^* is the inflation target, and that the policy rule $\phi\left(\right)$ is non-negative for all values of its arguments, is increasing in its first argument, and that $\partial\phi\left(\Pi_t^*, \Pi_t^*, \xi_t\right)/\partial\Pi_t > 1$, so that the "Taylor principle" applies around the inflation target. Suppose furthermore that agents know at some date $t_1 \geq t_0$, that the inflation target Π_t^* will remain constant at $\Pi_t^* = \Pi^* > \beta$ for all $t \geq t_1$. Then the model (5)–(12), (29) admits at least two possible steady states characterized by constant values $\bar{Y}_L, \bar{C}_L, \bar{l}_L, \bar{\imath}_L, \bar{\Pi}_L, \bar{W}_L$ as well as paths for the price levels $\bar{p}_t^{flex}, \bar{p}_t^{fix}, \bar{P}_t$ in all periods $t \geq t_1$.
(i) In the first (regular) steady state, the nominal interest rate $\bar{\imath}_L$ reaches a positive value $\bar{\imath}_L = \phi(\Pi_L^*, \Pi_L^*, \bar{\xi}_L) = \Pi_L^*\beta^{-1} - 1 > 0$, inflation is equal to the target rate $\bar{\Pi}_L = \Pi_L^*$, the price indices $\bar{p}_t^{flex}, \bar{p}_t^{fix}, \bar{P}_t$ grow at rate Π_L^*, and $\bar{Y}_L = \bar{l}_L = \bar{C}_L$ and $\bar{W}_L$ are given by

$$\frac{v_l(\bar{Y}_L)}{u_c(\bar{Y}_L)} = \frac{\theta - 1}{\theta} = \bar{W}_L. \tag{30}$$

(ii) In the second (Friedman) steady state, the nominal interest rate is at the ELB, $\bar{\imath}_L = 0$, the steady-state value $\bar{\Pi}_L$ shows perpetual deflation at the rate of time preference, $\bar{\Pi}_L = \beta$, the price indices (in log) fall at rate $\bar{\Pi}_L$, and $\bar{Y}_L = \bar{l}_L = \bar{C}_L$ and $\bar{W}_L$ are again given by (30).

Proof. See Appendix. ■

As stated, at least two long-run steady states are possible in this model: one with constant inflation at the central bank's target level and a positive nominal interest rate, and one with zero nominal interest rate (as in the Friedman rule) and perpetual deflation. Our earlier analysis took the first steady state as given. Importantly, however, the lack of existence of a medium run equilibrium does not depend on this assumption. As established in the next proposition, if the preference shock takes a low enough value in the medium run, then no medium run equilibrium exists in the non-linear version of the model under perfect foresight. We modify slightly Assumption A1 as follows:

A1' Consider the three periods $t = S, M, L$. In period $t = S$ there is an *unexpected* shock $\xi_S < \beta \bar{\Pi}_L^{-1} \bar{\xi}_L$, where $\bar{\xi}_L$ is the steady-state value of the preference shock and $\bar{\Pi}_L$ is the steady state inflation rate. In period $t = M$ the shock is still $\xi_M = \xi_S$. In periods $t \geq L$, the shock is back at its steady state $\bar{\xi}_L$. While the shock is unexpected in period $t = S$, there is perfect foresight between S, M, and L. Periods $t > L$ are identical to L: there are no shocks and agents have perfect foresight.

Proposition 9 *Under Assumption A1', for any given long-run equilibrium $\bar{Y}_L, \bar{C}_L, \bar{l}_L, \bar{\imath}_L, \bar{\Pi}_L, \bar{W}_L$ satisfying (5)–(29), there exists no medium-term equilibrium for small enough value of the exogenous disturbance ξ_M, i.e., if*

$$\xi_M < \beta \bar{\Pi}_L^{-1} \bar{\xi}_L. \tag{31}$$

Proof. See Appendix. ■

The condition (31) causing the absence of any medium-term equilibrium is analogous to the requirement that the natural rate of interest be low enough for the nominal rate to be constrained by the ELB, as seen in our analysis above. It requires ξ_M to be low enough, so that the representative consumer finds it preferable to consume less in the present (i.e., in the medium term) than in the future (i.e., the long run). Equivalently, the household prefers to save in the medium term. Doing so, reduces aggregate demand in the medium run, leading firms to lower their prices. This lowers inflation and increases the real interest rate. That increase in the medium-term real rate discourages households further from consuming, thereby leading to a collapse of the economy.

Note that if we are in the regular steady state with $\bar{\Pi}_L = \Pi_L^*$, (and $\beta \bar{\Pi}_L^{*-1} < 1$) then the higher the long-run inflation target Π_L^*, the easier it is for the medium-term equilibrium to exist. Indeed, the higher the long-run inflation target, the more difficult it is for the condition (31) to be satisfied, as ξ_M needs to fall possibly much below $\bar{\xi}_L$ for the medium run equilibrium to cease to exist. Instead, if we are in the "Friedman" steady state, where $\bar{\Pi}_L = \beta$, then the medium-run equilibrium ceases to exist and the economy collapses as soon as ξ_M falls below the long-run value of $\bar{\xi}_L$, regardless of the inflation target.

7 Non-Existence and Output Explosions in the New Keynesian Model

So far we limited our attention to a simple Phillips curve that nests a static Phillips Curve with the New Classical one to clearly illustrate the implications of money neutrality in the face of the ELB. In the last few decades, however, the New Keynesian Phillips curve has become more popular, both in quantitative monetary models and in discussions of monetary policy (see, e.g., Woodford (2003), Christiano et al. (2005), Smets and Wouters (2007)). The New Keynesian Phillips curve is derived from the same microfoundations as in Section 3.1, but under the assumption that constant fraction of randomly chosen firms reset their prices in every period. It is given by

$$\pi_t = \kappa \hat{Y}_t + \beta E_t \pi_{t+1}. \tag{32}$$

The IS equation, the policy rule and the ELB remain the same as before. This section confirms the basic insights from the previous analysis. The way non-existence appears in this model, is that if the shock that gives rise to the ELB has long enough duration, inflation and output drop without a bound. We establish this result considering two alternative shock processes common in the literature, first a deterministic one, and then a two-state Markov process with an absorbing state.

A3 There is an unexpected shock at time 0 that lasts until period T, so that $r_t^e = \underline{r} < \bar{r}$ for all $t = 0, 1, ..., T$, and $r_t^e = \bar{r}$ for all $t > T$.

A4 There is an unexpected shock at time 0 so that $r_0^e = \underline{r} < \bar{r}$. Conditional on $r_{t-1}^e = \underline{r}$, then, in every period $t > 0$ there is a fixed probability $1 - \mu$ that the shock reverts back to steady state $r_t = \bar{r}$. The steady state is an absorbing state. τ denotes the period in which the steady state is reached.

The next proposition summarizes the behavior of the model under A3.

Proposition 10 *Suppose A3 and that the inflation target is constant $\pi_t^* = \bar{\pi}^*$ for all $t > T$. We assume $\bar{r} > -\bar{\pi}^*$, $\phi_\pi > 1$. Moreover, we assume that $\underline{r}$ is sufficiently low for the lower bound to bind, that is $\underline{r} + \bar{\pi}^* < 0$, so that $i_t = 0$ for all $t = 0, 1, ...T$, and that the nominal interest rate is positive in all periods $t > T$. Then the model (14)–(15) and (32) implies a unique bounded equilibrium given by*

$$\pi_t = \begin{cases} -\underline{r} + c_1 (\beta e_1)^{T-t} + c_2 (\beta e_2)^{T-t} & \text{for } t = 0, 1, ...T \\ \bar{\pi}^* & \text{for } t > T \end{cases}$$

$$\hat{Y}_t = \begin{cases} c_0 + \frac{\sigma c_1}{\beta e_1 - 1} (\beta e_1)^{T-t} + \frac{\sigma c_2}{\beta e_2 - 1} (\beta e_2)^{T-t} & \text{for } t = 0, 1, ...T \\ \frac{1-\beta}{\kappa} \bar{\pi}^* & \text{for } t > T \end{cases}$$

where c_0, c_1, c_2, e_1, e_2 are constants satisfying $c_2 < 0$ and $0 < e_1 < 1 < \beta^{-1} < e_2$. In this equilibrium, output and inflation initially drop and then revert to the long-run steady state after date T. The magnitude of the initial drop in inflation and output grows exponentially with T, the duration of the natural rate of interest in its low state $\underline{r}$.

Proof. See Appendix. ■

While the equilibrium continues to exist under A3, longer durations of the ELB imply ever larger drops in output and inflation, given the dominant terms $c_2 (\beta e_2)^{T-t}$, so that these drops eventually become unboundedly large as $T - t$ becomes very large. Hence, the model imposes a limit on the duration of the recession to prevent it from exploding. Our proposition is closely related to the finding by Carlstrom, Fuerst and Paustian (2012), who show that an interest rate peg has a larger impact on inflation the longer the duration of the peg. It also relates to the literature on the "forward guidance puzzle" (Del Negro, Giannoni, Patterson, 2015; McKay, Nakamura and Steinsson, 2016) according to which expectations of future interest-rate changes have implausibly large effects on short-term output.

Assumption A4 allows for a useful graphical representation of the conditions for equilibrium existence in the New Keynesian model, which also connect to our earlier exposition. Let us call the stochastic period in which the shock recovers to its steady-state long-run value T. It is easy to confirm that the unique bounded solution for $t \geq T$ is $\pi_t = \bar{\pi}^*$ and $\hat{Y}_t = 0$, when the ELB is not binding in the long run. For periods $t < T$, however, $\underline{r} + \bar{\pi}^* < 0$ so that the ELB is binding, and setting for simplicity $\bar{\pi}^* = 0$, the model solves the following two equations:

$$\hat{Y}_t = \mu \tilde{E}_t \hat{Y}_{t+1} + \sigma\mu \tilde{E}_t \pi_{t+1} + \sigma \underline{r} \tag{33}$$

$$\pi_t = \kappa \hat{Y}_t + \mu\beta \tilde{E}_t \pi_{t+1}, \tag{34}$$

where we have used the solution for inflation and output in the long run to substitute out for conditional expectations; for example $E_t \hat{Y}_{t+1} = \mu \tilde{E}_t Y_{t+1} + (1-\mu) * 0$. The notation $\tilde{E}_t Y_{t+1}$ represents the conditional expectation of output conditional on the natural rate of interest being $\underline{r}$. The next proposition follows directly.

Proposition 11 *Suppose A4 and that the inflation target is constant at $\pi^* = 0$. Then $\pi_t = 0$ and $\hat{Y}_t = 0$ once the steady state is reached (for all $t \geq \tau$). If $(1-\mu)(1-\mu\beta) - \kappa\sigma\mu > 0$, there is a unique bounded solution in the short run, $t < \tau$, given by*

$$\begin{aligned}
\hat{Y}_S &= \frac{\sigma(1-\beta\mu)}{(1-\mu)(1-\mu\beta) - \sigma\kappa\mu} r_S^e \\
\pi_S &= \frac{\sigma\kappa}{(1-\mu)(1-\beta\mu) - \sigma\kappa\mu} r_S^e.
\end{aligned}$$

The proof of this proposition is similar to that of Proposition 3 in Eggertsson (2010). It establishes that a necessary and sufficient condition for obtaining a unique bounded solution is $(1-\mu)(1-\mu\beta) - \kappa\sigma\mu > 0$. We provide below an intuitive explanation for this condition using a graphical analysis closely resembling what we have done so far in the paper.

Conditional on there being a unique bounded solution to the system, the solution is a pair of numbers $(\hat{Y}_S, \pi_S)$ that solve (33) and (34), which can be expressed as

$$\hat{Y}_S^{AD} = \frac{\sigma\mu}{1-\mu}\pi_S + \frac{\sigma}{1-\mu}\underline{r} \tag{35}$$

$$\pi_S = \frac{\kappa}{1-\mu\beta}\hat{Y}_S^{AS}. \tag{36}$$

These equations represent the AD and the AS relationships which are plotted in Figure 12. At the intersection point A, $\mu = 0$, i.e., the shock is expected to revert back to steady state with probability 1 next period. Output is completely demand determined and is equal to $\sigma\underline{r}$. The AD is vertical in that case, as it doesn't depend on inflation. The Phillips curve (or upward-sloping AS) then determines the inflation rate associated with the given level of output. At this point, the trade-off between inflation and output is given by κ. Consider now an increase in μ, which has the effect of increasing the expected duration of the ELB. This shifts the AD curve so that inflation becomes positively related to output. This is because a fall in inflation in the short run will be reflected in lower inflation expectations, thus increasing the real interest rate and suppressing demand. In addition, the higher μ also results in a shift in the AS curve towards a steeper curve, which is closer monetary neutrality. As a result, the higher μ leads to a larger

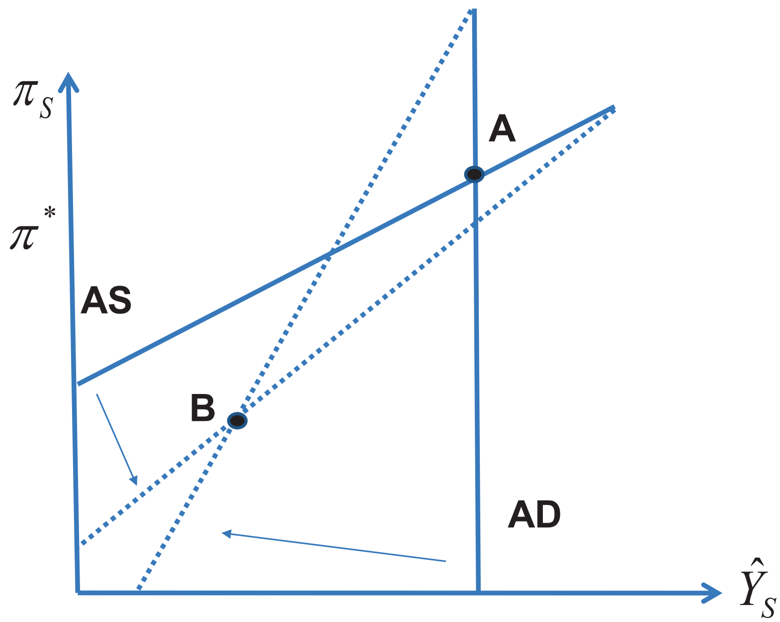

Figure 12: Aggregate supply and demand equations in the New Keynesian model with the shock specified in Assumption 4.

output contraction and inflation decline, at point B. In all, the shift of the Phillips curve towards a steeper AS curve featuring more monetary neutrality intensifies the economic contraction at the ELB.

Why does an increase in the duration of ELB move the models closer to monetary neutrality? If $\beta = 1$, then in the limit when $\mu = 1$ there is full monetary neutrality. The intuition is that the new Keynesian Phillips curve has forward looking expectations, so that the longer the ELB is expected to bind, the closer the AS approaches the AS that would obtain in steady state, as can be seen in (36).

Figure 12 reveals another important observation, which is closely related to the previous proposition. As μ increases, then the two curves become closer and closer to parallel, leading to lower and lower levels of output and inflation. Both drop without bound when the AD and AS lines are parallel, and no equilibrium exists, similarly to the our earlier cases discussed. The condition in the last proposition for a unique bounded solution is equivalent to the requirement that the aggregate demand curve be steeper than aggregate supply, i.e. $\frac{\sigma\mu}{1-\mu} > \frac{1-\mu\beta}{\kappa}$.[10]

A final observation, although we do not show the explicit formulas, is that in this model as in the preceding discussion, when the output collapses, the benefit of inflation becomes unbounded.

[10] Observe that if this condition holds, the log-linear model exhibits indeterminacy. As shown by Eggertsson and Singh (2019), this region of the parameter space does not represent a valid solution in the non-linear version of the model. Intuitively, as explained in the text, as the duration of the crisis increases, the drop in output ultimately becomes unbounded, which results in the linear approximation to be no longer valid.

8 Equilibrium Non-Existence in a Secular Stagnation

There is a recent literature on long-lasting monetary slumps, often identified with secular stagnation. Here we stress that this literature needs to assume monetary non-neutrality to establish existence of equilibria for the same reason as we have documented, i.e. that the aggregate demand side will otherwise clash with the supply side of the model, yielding no equilibria. This literature on long demand slumps highlights that the aggregate demand side of the model needs to be extended to consider long-lasting drops in the interest rate. Common modeling devices are overlapping generations models (Eggertsson, Mehrotra, Robbins, 2019), wealth in utility (Michaillat and Saez, 2019) or incomplete asset markets. These extensions can often be summarized by the linearized version of the IS curve with discounting

$$\hat{Y}_t = \delta E_t \hat{Y}_{t+1} - \sigma(i_t - E_t \pi_{t+1} - r_t^e) \tag{37}$$

where $0 < \delta < 1$. A key property of this equation, is that in steady state

$$\hat{Y}_L = -\frac{1}{1-\delta}\sigma(i_L - \pi_L - r_L^e), \tag{38}$$

that is, the steady-state long-run demand is decreasing in the real interest rate. A key to obtaining a downward sloping relationship between long-run demand and the interest rate is $\delta < 1$.

Related to this extension, many authors have pointed out that assuming $\delta = 1$ is responsible for the contractionary explosive behavior of output the New Keynesian model, which has some unattractive properties such as, for example, giving rise to the forward guidance puzzle (see Del Negro, Giannoni and Patterson, 2012; McKay, Nakamura and Steinsson, 2016). A key resolution of this literature has been to include discounting in the IS equation as in shown in (37).

What has not been emphasized as much, however, is the role of monetary neutrality emphasized in this paper. If the AS equation has too strong a monetary neutrality, then the model exhibits explosive behavior as well, or gives rise to equilibrium non-existence. The easiest way to see this is to complement the equation (38) with an aggregate supply. It is easy to see that if we assume the New Classical Phillips Curve (1) with $\lambda = 1$, then once again we have equilibrium non-existence, exactly as we have documented in this paper, regardless of the value of δ. Meanwhile, in the NK model, the consider the long-run trade-off is given by

$$\hat{Y}_L = \frac{1-\beta}{\kappa}\pi_L. \tag{39}$$

As β approaches 1, then the model approaches money neutrality. Assuming $r_L^e < 0$ so that the ELB is binding, we obtain the following solution for long run output

$$\hat{Y}_L = \frac{\sigma}{1-\delta-\frac{\sigma\kappa}{1-\beta}} r_L^e.$$

The condition for equilibrium existence in the linearized model is: $1-\delta > \frac{\sigma\kappa}{1-\beta}$.[11] We see that as β approaches 1, and the model moves closer and closer to monetary neutrality, the term $\frac{\sigma\kappa}{1-\beta}$ increases, and output becomes more and more negative until its contraction becomes unbounded and the equilibrium ceases to exist.

[11] See Eggertsson and Singh (2019) for a discussion in a closely related context for why this is required for equilibrium existence. Essentially what happens is that the model "explodes" as $1-\delta-\frac{\sigma\kappa}{1-\beta}$ approaches zero, and the linear approximation is no longer valid when $1-\delta-\frac{\sigma\kappa}{1-\beta}$ takes on negative values. In the non-linear counterpart of the model, there is no longer an intersection between aggregate supply and demand in this case.

In calibrations of the New Keynesian model, the parameter β is typically assumed to be very close to 1, e.g. 0.99 when calibrated in quarterly frequencies. It is for this reason that authors studying questions related to secular stagnation typically adopt a Phillips Curve specification with a permanent trade-off between inflation and output, for example due to wages being rigid downward. Examples in this vein include Eggertsson, Mehrotra and Robbins (2019) and Schmitt-Grohé and Uribe (2017). A notable exception is Michaillat and Saez (2019), who study long-run equilibria in a New Keynesian setting. They obtain long-run trade-offs by assuming a low value of β; appealing to field and laboratory evidence they argue for a annual discount rate as low as $\beta = 0.9$ at quarterly frequencies.

9 Flexible Prices and Equilibrium Non-Existence

In the introduction we noted a connection to the price flexibility paradox and suggested that a key observation is that there is no equilibrium if prices are flexible and the central bank targets zero inflation. We now develop this argument further. Consider a flexible price economy, given by a Fisher Equation and the ELB

$$\begin{aligned} r_t &= i_t - E_t \pi_{t+1} \\ i_t &\geq 0. \end{aligned}$$

Because this economy has flexible prices the real interest rate r_t is exogenous, while the nominal interest rate i_t is controlled by the central bank. This economy is for example derived in Woodford (2003), corresponding to a classic endowment economy. Consider a shock to the real interest rate so that it is negative in period 0, $r_0 < 0$, and $r_t = \beta^{-1} - 1 > 0$ for all $t > 0$. As the ELB is no longer binding from period 1 on, the central bank can generate any inflation rate it chooses. How does it implement this inflation rate? For concreteness, one can for example imagine a reduced-form money demand function as in Krugman (1998)

$$\frac{M_t}{P_t} \geq Y_t$$

which is slack when the ELB is binding. Since the interest rate is positive in period $t \geq 1$, nothing prevents the central bank from meeting its assumed objective and choosing a money supply such that the price level in period 1 and onward is the same as in period 0, i.e., $P_t = P_0$ for all $t > 0$, so that $\pi_t = 0$ for all $t > 0$. This implies however that the equilibrium cannot exist in period 0

$$r_0 = i_0 - \pi_1 = i_0 \geq 0,$$

which contradicts our initial assumption that $r_0 < 0$. The flexible-price economy needs to generate negative real interest rates in period 0, and the only way it can do so is via expected inflation. Yet π_1 is ultimately determined by the central bank in the next period. There is no guarantee that the central bank will deliver the required inflation given that it is assumed to target zero inflation. There is also no reason for it not to pin down the price level at that time, for example via money supply, since the interest rate is then again positive.

10 Non-Existence and Alternative Monetary Policy Specifications

In this paper we have focused on the equilibrium non-existence under the assumption that the central bank targets some inflation target π_t^* whenever it can, with the only reason preventing it from doing so being the effective lower bound. This does not mean, however, that non-existence of equilibria arises under all policy specifications. Indeed, the equilibrium can be restored and the ELB can be made irrelevant if the central bank chooses for example a sufficiently high inflation target. Alternatively, the equilibrium could be restored even with a low inflation target, if the central bank seeks to stabilize the price level. To see this, consider again the example discussed in the previous section of a flexible-price economy where the central bank targets zero inflation, but suppose now that the central bank targets a price level $\bar{p}$. In this case

$$r_0 = i_0 - \pi_1 = -p_1 + p_0 = -\bar{p} + p_0$$

or $p_0 = r_0 + \bar{p}$. In period 0, the price level falls sufficiently so as to generate enough expected inflation. More generally, price-level targeting regimes tend to be more robust for generating a stable equilibrium, and are similarly less prone to generating equilibrium non-existence. The optimal commitment policy is a policy regime that is relatively close to a price-level targeting regime, as shown in Eggertsson and Woodford (2003), Giannoni and Woodford (2005), and Giannoni (2014). The same can be said about certain inertial Taylor-type interest-rate rules that assign enough weight to lagged interest rates.

11 Conclusion

Conventional wisdom suggests that medium-term money neutrality imposes strong limitations on the effects of monetary policy. In particular, as expectations become more ingrained it is commonly believed that a monetary stimulus tends to have smaller output effects and results in higher inflation. As we have shown in this paper, this medium-term monetary neutrality assumption runs into a direct conflict with the aggregate demand side of modern general equilibrium models once the effective lower bound on the nominal interest rate is taken into account. This results in some cases in non-existence of equilibria, or in output contracting without a bound at the ELB — a phenomenon we term contractionary black holes. Our analysis first discussed the New Classical Phillips curve as it features full monetary neutrality in the medium run, and considered deviations from this benchmark, which produce a medium-run trade-off between inflation and output. A key conclusion is that the closer the aggregate supply is to exhibiting medium-term money neutrality, the more likely the model is to produce equilibrium non-existence or a very large output contraction in the short run, at the ELB. This is because an adverse shock at the ELB causes a decline in short-run inflation and in medium-term inflation expectations, which increases the real interest rate and reinforces the output contraction. The closer we are to money neutrality, the larger is the decline in medium-run inflation expectations, the higher the increase the real interest rate and thus the sharper the output contraction in the short run. Our analysis then proceeded with the New Keynesian Phillips curve. In that case, while the result manifests itself via the duration at the ELB, we found again that the closer the Phillips curve is to producing medium-term monetary neutrality, the sharper is the short-run output contraction at the ELB.

Paradoxically, the case for expansionary monetary policy at the ELB is even stronger in models that feature near money neutrality. It is in this case that a monetary stimulus most effectively raises medium-term inflation expectations and lowers real interest rates, so as to bring output closer to its potential. Our analysis reveals that in the face of the ELB, output and inflation can be stabilized by raising inflation expectations in a relatively short period after the shock which brought the economy to the ELB has subsided. However, as in conventional analyses, once the economy has exited from the ELB, there is no meaningful gain from raising the long-run inflation target.

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A Appendix: Linearization of the Optimal Pricing Conditions

This sections shows how a linearization of the optimal pricing conditions (9)-(12) results in the Phillips curve (1) or equivalently (13). Using (7) to replace W_t in (9)-(10), and linearizing the resulting expressions yields

$$\begin{aligned} \hat{p}_t^{flex} &= \hat{P}_t + (\omega + \sigma^{-1})\hat{Y}_t \\ \hat{p}_t^{fix} &= E_{t-1}\left[\hat{P}_t + (\omega + \sigma^{-1})\hat{Y}_t\right] = E_{t-1}\hat{p}_t^{flex}, \end{aligned} \tag{40}$$

where $\omega \equiv \frac{v_{ll}l}{v_l} > 0$ and $\sigma^{-1} \equiv -\frac{u_{cc}C}{u_c} > 0$. Linearizing in turn (12) yields

$$\hat{P}_t = (1 - \gamma_{fix} - \gamma_{ind})\hat{p}_t^{flex} + \gamma_{fix}\hat{p}_t^{fix} + \gamma_{ind}\hat{P}_{t-1}. \tag{41}$$

It follows that

$$E_{t-1}\hat{P}_t = (1 - \gamma_{fix} - \gamma_{ind})\hat{p}_t^{fix} + \gamma_{fix}\hat{p}_t^{fix} + \gamma_{ind}\hat{P}_{t-1} = (1 - \gamma_{ind})\hat{p}_t^{fix} + \gamma_{ind}\hat{P}_{t-1},$$

and thus that

$$\hat{P}_t - E_{t-1}\hat{P}_t = \pi_t - E_{t-1}\pi_t = (1 - \gamma_{fix} - \gamma_{ind})\left(\hat{p}_t^{flex} - \hat{p}_t^{fix}\right).$$

Using (41) to solve for $\hat{p}_t^{fix}$, we can rewrite the previous equation as

$$\begin{aligned} \pi_t - E_{t-1}\pi_t &= (1 - \gamma_{fix} - \gamma_{ind})\left(\hat{p}_t^{flex} - \frac{1}{\gamma_{fix}}\hat{P}_t + (\frac{1 - \gamma_{fix} - \gamma_{ind}}{\gamma_{fix}})\hat{p}_t^{flex} + \frac{\gamma_{ind}}{\gamma_{fix}}\hat{P}_{t-1}\right) \\ &= (1 - \gamma_{fix} - \gamma_{ind})\frac{1 - \gamma_{ind}}{\gamma_{fix}}(\hat{p}_t^{flex} - \hat{P}_t) - (1 - \gamma_{fix} - \gamma_{ind})\frac{\gamma_{ind}}{\gamma_{fix}}\pi_{t-1}. \end{aligned}$$

Using (40) to eliminate $\hat{p}_t^{flex}$ yields finally the Phillips curve (1), with $\kappa \equiv \frac{(1-\gamma_{fix}-\gamma_{ind})\frac{1-\gamma_{ind}}{\gamma_{fix}}}{\left(1+(1-\gamma_{fix}-\gamma_{ind})\frac{\gamma_{ind}}{\gamma_{fix}}\right)}\left(\omega + \sigma^{-1}\right) > 0$, $\lambda \equiv \left(1 + (1 - \gamma_{fix} - \gamma_{ind})\frac{\gamma_{ind}}{\gamma_{fix}}\right)^{-1} \in [0, 1]$.

B Appendix: Proof of Propositions for the New Classical Model

B.1 Proof of Proposition 2

Proof. Under the assumptions A1, $\bar{r} > -\pi_L^*$, $\phi_\pi > 1$ and that the nominal interest rate is always positive in the long run, it follows from Proposition 1 that a long-run equilibrium is well defined with $\hat{Y}_L = 0$, and $E_M\pi_L = \pi_L^*$. Next, suppose, as a way of contradiction, that a medium run equilibrium $\left\{\pi_M, \hat{Y}_M, i_M\right\}$ satisfying (1) and (14)–(15) exists. Perfect foresight between periods S and M implies $E_S\pi_M = \pi_M$, so that the aggregate supply equation (1) in period M implies $\hat{Y}_M = 0$. On the demand side, consider two alternative cases. First, if $r_M^e + \pi_M^* + \phi_\pi(\pi_M - \pi_M^*) \leq 0$, then $i_M = 0$ by (15). Since $r_M^e + \pi_L^* < 0$ by assumption, it follows from (14) that $\hat{Y}_M = \sigma\left(\pi_L^* + r_M^e\right) < 0$. Alternatively, if $r_M^e + \pi_M^* + \phi_\pi(\pi_M - \pi_M^*) > 0$, then $i_M > 0$ by (15). In this case, (14) implies

$$\hat{Y}_M = -\sigma\left(i_M - \pi_L^* - r_M^e\right) = -\sigma i_M + \sigma\left(\pi_L^* + r_M^e\right) < 0.$$

So in both cases (14)–(15) imply that $\hat{Y}_M < 0$. This leads to a contradiction with the implication of (1) according to which $\hat{Y}_M = 0$. Hence no medium-run equilibrium $\left\{\pi_M, \hat{Y}_M, i_M\right\}$ satisfying (1) and (14)–(15) exists. ∎

B.2 Proof of Proposition 5

Proof. Under the assumptions A2, $\bar{r} > -\pi_L^*$, $\phi_\pi > 1$ and that the nominal interest rate is always positive in the long run, it follows from Proposition 1 that $\hat{Y}_L = 0$. Using this, equations (21)–(26) can be written in matrix form as

$$\begin{bmatrix} \alpha & -\alpha & -\kappa & 0 & 0 & 0 & 0 & 0 \\ -(1-\alpha) & 1-\alpha & 0 & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\phi_\pi & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\kappa \\ -\sigma(1-\alpha) & -\sigma\alpha & -(1-\alpha) & -\alpha & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi_M^{low} \\ \pi_M^{high} \\ \hat{Y}_M^{low} \\ \hat{Y}_M^{high} \\ i_M^{low} \\ i_M^{high} \\ \pi_S \\ \hat{Y}_S \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma(\pi_L^* + r_S^e) \\ \sigma(\pi_L^* + r_L^e) \\ 0 \\ r_L^e - (\phi_\pi - 1)\pi_M^* \\ 0 \\ \sigma r_S^e \end{bmatrix}$$

We note that the determinant of the matrix on the left is $\alpha\kappa\sigma\phi_\pi$. So, provided that $\alpha > 0$ and that $\kappa\sigma\phi_\pi > 0$, we can invert the matrix to obtain the solution expressed in the proposition. ■

C Appendix: Nonlinear Model

C.1 Model

The nonlinear model described in the text is repeated here for convenience using $Y_t = l_t = C_t$ to eliminate C_t and l_t. The household's optimal conditions for consumption and leisure are given by

$$u_c(Y_t)\xi_t = (1+i_t)\beta E_t\left[u_c(Y_{t+1})\xi_{t+1}\Pi_{t+1}^{-1}\right] \tag{42}$$

$$W_t = \frac{v_l(Y_t)}{u_c(Y_t)} \tag{43}$$

where

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} \tag{44}$$

and the nominal interest rate satisfies the ELB constraint

$$i_t \geq 0. \tag{45}$$

The firms' optimal conditions for pricing, in the case that $\gamma_{ind} = 0$ (corresponding to the New Classical benchmark), are given by

$$\frac{p_t^{flex}}{P_t} = \frac{\theta}{\theta-1}W_t \tag{46}$$

and

$$E_{t-1}\left[u_c(Y_t)Y_t\left(\frac{p_t^{fix}}{P_t}\right)^{-\theta}\left(\frac{p_t^{fix}}{P_t} - \frac{\theta}{\theta-1}W_t\right)\right] = 0, \tag{47}$$

and the aggregate price level satisfies

$$1 = (1-\gamma_{fix})\left(\frac{p_t^{flex}}{P_t}\right)^{1-\theta} + \gamma_{fix}\left(\frac{p_t^{fix}}{P_t}\right)^{1-\theta}. \tag{48}$$

Finally government policy is given by an interest rate reaction function

$$i_t = \phi(\Pi_t, \Pi_t^*, \xi_t) \tag{49}$$

where we assume that the function $\phi\,()$ is non-negative for all values of its arguments (otherwise policy would not be feasible given the ELB) and increasing in Π_t. In addition, fiscal policy is assumed to be such that (5) is satisfied. An equilibrium can now be defined as a set of stochastic processes $\{\frac{p_t^{flex}}{P_t}, \frac{p_t^{fix}}{P_t}, \Pi_t, Y_t, W_t, i_t\}$ and a fiscal policy that satisfy equations (5), (42)–(49) in all periods $t \geq t_0$, for given $\{\xi_t, \Pi_t^*\}$.

C.2 Long-run steady state: Proof of Proposition 8

Proof. Consider the long-run steady states given by constant values $\bar{Y}_L, \bar{\imath}_L, \bar{\Pi}_L, \bar{W}_L$, as well as paths for the price levels $\bar{p}_t^{flex}, \bar{p}_t^{fix}, \bar{P}_t$ for given exogenous variables $\bar{\xi}_L$, $\bar{\Pi}_L^*$ that satisfy the above equations in all periods $t \geq t_1$. Optimal pricing conditions for all $t \geq t_1$ are given by

$$\frac{\bar{p}_t^{flex}}{\bar{P}_t} = \frac{\theta}{\theta - 1}\bar{W}_L$$

and

$$\frac{\bar{p}_t^{fix}}{\bar{P}_t} = \frac{\theta}{\theta - 1}\bar{W}_L$$

given that $u_c\left(\bar{Y}_L\right)\bar{Y}_L\left(\bar{p}_t^{fix}/\bar{P}_t\right)^{-\theta} > 0$ by assumption. The two optimal pricing equations combined with the aggregate price level equation (48)

$$1 = (1 - \gamma_{fix})\left(\frac{\bar{p}_t^{flex}}{\bar{P}_t}\right)^{1-\theta} + \gamma_{fix}\left(\frac{\bar{p}_t^{fix}}{\bar{P}_t}\right)^{1-\theta}$$

imply

$$\bar{P}_t = \bar{p}_t^{flex} = \bar{p}_t^{fix}$$

and

$$\bar{W}_L = \frac{\theta - 1}{\theta}.$$

Next, combining this with (43) determines implicitly steady-state output using

$$\frac{v_l(\bar{Y}_L)}{u_c(\bar{Y}_L)} = \frac{\theta - 1}{\theta}. \tag{50}$$

To determine steady-state inflation, we use the consumption Euler equation (42) at the steady state

$$u_c(\bar{Y}_L)\bar{\xi}_L = (1 + \bar{\imath}_t)\beta u_c(\bar{Y}_L)\bar{\xi}_L\bar{\Pi}_{t+1}^{-1}$$

for any $t \geq t_1$. This simplifies to

$$\beta^{-1} = (1 + \bar{\imath}_t)\bar{\Pi}_{t+1}^{-1}. \tag{51}$$

Combining this with the government's interest-rate reaction function

$$\bar{\imath}_t = \phi(\bar{\Pi}_t, \bar{\Pi}_t^*, \bar{\xi}_t)$$

we get

$$\bar{\Pi}_{t+1} = \beta(1 + \phi(\bar{\Pi}_t, \bar{\Pi}_t^*, \bar{\xi}_t)), \tag{52}$$

for any $t \geq t_1$. As analyzed in Benhabib, Schmitt-Grohé and Uribe (2001), this equation admits at least two constant solutions. One solution $\bar{\Pi}_t = \bar{\Pi}_t^*$, and $\bar{\imath}_t = \phi(\bar{\Pi}_t^*, \bar{\Pi}_t^*, \bar{\xi}_t) = \beta^{-1}\bar{\Pi}_t^* - 1 > 0$, for all $t \geq t_1$. The other constant solution is $\bar{\Pi}_t = \beta < 1$ and $\bar{\imath}_t = \phi\left(\beta, \bar{\Pi}_t^*, \bar{\xi}_t\right) = 0$, for all $t \geq t_1$. As shown in Woodford (2003, chap. 2), for any initial inflation $\Pi_0 \in \left(0, \bar{\Pi}_L^*\right)$, one of these steady states will eventually be reached. In contrast, for any $\Pi_0 > \bar{\Pi}_L^*$, inflation will increase forever and get unboundedly large. Such an equilibrium is not consistent with a constant steady state inflation. ■

C.3 Non-existence of medium term equilibrium: Proof of Proposition 9

Proof. Given Assumption 1', there is perfect foresight between periods M and L, so that the optimal pricing conditions (46)–(47) simplify to

$$\frac{p_M^{flex}}{P_M} = \frac{\theta}{\theta - 1} W_M$$

and

$$\frac{p_M^{fix}}{P_M} = \frac{\theta}{\theta - 1} W_M,$$

given that $u_c\left(Y_M\right) Y_M \left(p_M^{fix}/P_M\right)^{-\theta} > 0$ by assumption. Combining this with (48) yields

$$\begin{aligned} P_M &= p_M^{flex} = p_M^{fix} \\ W_M &= \frac{\theta - 1}{\theta}. \end{aligned}$$

Using (43), we can determine medium-term output supplied

$$\frac{v_l(Y_M)}{u_c(Y_M)} = \frac{\theta - 1}{\theta},$$

so that

$$Y_M = \bar{Y}_L. \tag{53}$$

On the demand side, however, equilibrium output must satisfy (42) or

$$u_c(Y_M)\xi_M = (1 + i_M)\beta u_c(\bar{Y}_L)\bar{\xi}_L\bar{\Pi}_L^{-1}.$$

Using (53), this simplifies to

$$\xi_M = (1 + i_M)\beta\bar{\xi}_L\bar{\Pi}_L^{-1}$$

so that $i_M < 0$, if condition (31) holds. It follows that the ELB condition (45) is violated and hence that there is no medium-term equilibrium if (31) holds. ■

D Appendix: Proof of Propositions for the New Keynesian Model

D.1 Proof of Proposition 10

We start by characterizing the long-run steady state. We next describe the equilibrium after period T and finish with a description of the equilibrium for dates $t = 0, 1, ...T$.

Long-run steady state. The model given by equations (14)–(15) and (32) admits a long-run steady state $\bar{\pi}, \bar{Y}, \bar{\imath}, \bar{r}^e$ that satisfies

$$\begin{aligned} \bar{\imath} &= \bar{\pi} + \bar{r}. \\ \bar{\imath} &= \max\{0, \bar{r} + \bar{\pi}^* + \phi_\pi(\bar{\pi} - \bar{\pi}^*)\} \\ (1-\beta)\,\bar{\pi} &= \kappa \bar{Y} \end{aligned}$$

Combining the first two equations and using the assumption $\bar{\imath} > 0$, we obtain

$$\bar{\pi} + \bar{r} = \bar{r} + \bar{\pi}^* + \phi_\pi(\bar{\pi} - \bar{\pi}^*)$$

which simplifies to $(\phi_\pi - 1)(\bar{\pi} - \bar{\pi}^*) = 0$, and hence to $\bar{\pi} = \bar{\pi}^*$, since $\phi_\pi > 1$. It follows that the steady-state level of output $(\hat{Y}_t)$ is given by $\bar{Y} = \frac{1-\beta}{\kappa}\bar{\pi}^*$.

Equilibrium after period T. After period T, the equations (14)–(15) and (32) reduce to

$$\begin{aligned} \pi_t &= \kappa \hat{Y}_t + \beta E_t \pi_{t+1} \\ \hat{Y}_t &= E_t \hat{Y}_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \bar{r}) \\ i_t &= \bar{r} + \bar{\pi}^* + \phi_\pi(\pi_t - \bar{\pi}^*), \end{aligned}$$

using the assumption $i_t \geq 0$ for all $t > T$. Combining the last two equations to eliminate i_t, we can rewrite the system in matrix form as

$$\begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix} E_t \begin{bmatrix} \pi_{t+1} \\ \hat{Y}_{t+1} \end{bmatrix} = \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \sigma\phi_\pi & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ \hat{Y}_t \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma(1-\phi_\pi)\,\bar{\pi}^* \end{bmatrix}$$

or

$$E_t z_{t+1} = A z_t + \delta \tag{54}$$

where

$$z_t \equiv \begin{bmatrix} \pi_t \\ \hat{Y}_t \end{bmatrix}, \qquad \delta \equiv \begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \sigma(1-\phi_\pi)\,\pi_t^* \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma(1-\phi_\pi)\,\bar{\pi}^* \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix}^{-1} \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \sigma\phi_\pi & 1 \end{bmatrix} = \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \sigma\left(\phi_\pi - \beta^{-1}\right) & 1 + \beta^{-1}\kappa\sigma \end{bmatrix}.$$

Note that with $\phi_\pi > 1$, we have $\det(A) = \beta^{-1}(\kappa\sigma\phi_\pi + 1) > 1$, and $tr\,(A) = \beta^{-1} + \beta^{-1}\kappa\sigma + 1$. This implies: $\det(A) - tr\,(A) = -1 + \beta^{-1}\kappa\sigma(\phi_\pi - 1) > -1$, and $\det(A) + tr\,(A) = \frac{1}{\beta}(\beta + \kappa\sigma(\phi_\pi + 1) + 2) > 1$. It then follows from Proposition C.1. in Woodford (2003, p. 670) that the matrix A has both eigenvalues outside the unit circle, so that A^{-1} exists and has both eigenvalues inside the unit circle.

Iterating forward equation (54), we have

$$\begin{aligned} z_t &= -A^{-1}\delta_t + E_t A^{-1} z_{t+1} \\ &= E_t\left(-A^{-1}\delta - A^{-2}\delta ... - A^{-n}\delta + A^{-n} z_{t+n}\right). \end{aligned}$$

Since $\lim_{n\to\infty} E_t A^{-n} z_{t+n} = 0$ for any bounded process $\{z_t\}$, we obtain the unique bounded solution

$$z_t = -\sum_{T=t}^{\infty} A^{-(T-t+1)}\delta = -A^{-1}\left(I - A^{-1}\right)^{-1}\delta = (I-A)^{-1}\delta = \begin{bmatrix} \bar{\pi}^* \\ \frac{1-\beta}{\kappa}\bar{\pi}^* \end{bmatrix},$$

so that inflation and output are at their long-run steady state at all dates $t > T$.

Equilibrium in periods $t = 0, 1, ..., T.$ With the ELB assumed binding in periods $t = 0, 1, ..., T$, the model reduces to

$$\begin{aligned} \pi_t &= \kappa \hat{Y}_t + \beta E_t \pi_{t+1} \\ \hat{Y}_t &= E_t \hat{Y}_{t+1} + \sigma(E_t \pi_{t+1} + \underline{r}). \end{aligned} \tag{55}$$

Combining the last two equations yields

$$E_t \left[\pi_t - (1 + \beta + \kappa\sigma) \pi_{t+1} + \beta \pi_{t+2}\right] = \kappa \sigma \underline{r}. \tag{56}$$

Let $\underline{\pi}$ be the "steady state" value of π_t, in periods $t = 0, ...T$. It satisfies $\underline{\pi} - (1 + \beta + \kappa\sigma) \underline{\pi} + \beta \underline{\pi} = \kappa \sigma \underline{r}$, or $\underline{\pi} = -\underline{r}$. We can then rewrite (56) as the homogenous equation

$$E_t \left[B(L) \tilde{\pi}_{t+2}\right] = 0 \tag{57}$$

where $\tilde{\pi}_t \equiv \pi_t - \underline{\pi}$, the lag polynomial $B(L)$ is given by

$$\begin{aligned} B(L) &\equiv L^2 - (1 + \beta + \kappa\sigma) L + \beta \\ &= \beta (1 - e_1 L) (1 - e_2 L) \end{aligned} \tag{58}$$

and e_1, e_2 are the two roots of the characteristic polynomial $P(x) = \beta x^2 - (1 + \beta + \kappa\sigma) x + 1$. Note that $P(x)$ is convex, and $P(0) = 1 > 0$, $P(1) = -\kappa\sigma < 0, P\left(\beta^{-1}\right) = -\kappa\sigma\beta^{-1} < 0$, so that $P(x) = 0$ admits two real solutions $0 < e_1 < 1 < \beta^{-1} < e_2$. Expanding (58) and comparing it to $B(L)$ reveals that $e_1 e_2 = \beta^{-1}$. Using this, we can rewrite (57) as

$$0 = E_t \left[(1 - e_1 L) (1 - e_2 L) \tilde{\pi}_{t+2}\right]$$

Defining $z_t \equiv (1 - e_1 L) \tilde{\pi}_t$, this can be expressed as

$$0 = E_t \left[-e_2^{-1} L^{-1} L (1 - e_2 L) z_{t+2}\right] = E_t \left[\left(1 - e_2^{-1} L^{-1}\right) z_{t+1}\right]$$

or

$$\begin{aligned} E_t z_{t+1} &= E_t \left[e_2^{-1} z_{t+2}\right] \\ &= E_t \left[e_2^{-(T-t-1)} z_T\right], \end{aligned}$$

where the last equality is obtained after iterating forward. Using again the definition of z_t, we can then write

$$E_t \left[(1 - e_1 L) \tilde{\pi}_{t+1}\right] = E_t \left[e_2^{-(T-t-1)} \left(\tilde{\pi}_T - e_1 \tilde{\pi}_{T-1}\right)\right]$$

or

$$E_t \left[\tilde{\pi}_{t+1} - e_1 \tilde{\pi}_t\right] = E_t \left[e_2^{-(T-t-1)} \left(\tilde{\pi}_T - e_1 \tilde{\pi}_{T-1}\right)\right].$$

This expression can be iterated forward to yield

$$\begin{aligned} \tilde{\pi}_t &= E_t \left[-e_1^{-1} e_2^{-(T-t-1)} \left(\tilde{\pi}_T - e_1 \tilde{\pi}_{T-1}\right) + e_1^{-1} \tilde{\pi}_{t+1}\right] \\ &= E_t \left[-e_1^{-(T-t)} \left(e_1^{T-t-1} e_2^{-(T-t-1)} + e_1^{T-t-2} e_2^{-(T-t-2)} + e_1^{T-t-3} e_2^{-(T-t-3)} + ... + e_1^0 e_2^0\right) \left(\tilde{\pi}_T - e_1 \tilde{\pi}_{T-1}\right) + e_1^{-(T-t)} \tilde{\pi}_T\right] \\ &= e_1^{-(T-t)} E_t \left[-\frac{1 - \left(e_1 e_2^{-1}\right)^{T-t}}{1 - \left(e_1 e_2^{-1}\right)} \left(\tilde{\pi}_T - e_1 \tilde{\pi}_{T-1}\right) + \tilde{\pi}_T\right]. \end{aligned}$$

Note that (56) implies

$$\begin{aligned}
\pi_T &= (1+\kappa\sigma)\,\bar{\pi}^* + \kappa\sigma\underline{r} \\
\tilde{\pi}_T &= \pi_T - \underline{\pi} = (1+\kappa\sigma)\,(\bar{\pi}^* + \underline{r})
\end{aligned}$$

and

$$\begin{aligned}
\pi_{T-1} &= (1+\beta+\kappa\sigma)\,\pi_T - \beta\bar{\pi}^* + \kappa\sigma\underline{r} = \left((1+\kappa\sigma)^2 + \beta\kappa\sigma\right)\bar{\pi}^* + (2+\beta+\kappa\sigma)\,\kappa\sigma\underline{r} \\
\tilde{\pi}_{T-1} &= \pi_{T-1} - \underline{\pi} = \left((1+\kappa\sigma)^2 + \beta\kappa\sigma\right)(\bar{\pi}^* + \underline{r})
\end{aligned}$$

It follows, using $\beta e_1 e_2 = 1$, that

$$\begin{aligned}
\tilde{\pi}_t &= e_1^{-(T-t)}\left[-\frac{1-\left(e_1e_2^{-1}\right)^{T-t}}{1-\left(e_1e_2^{-1}\right)}\left((1+\kappa\sigma) - e_1\left((1+\kappa\sigma)^2+\beta\kappa\sigma\right)\right) + (1+\kappa\sigma)\right](\bar{\pi}^*+\underline{r}) \\
&= \left[\frac{(\beta e_1)^{T-t} - (\beta e_2)^{T-t}}{1-\left(e_1e_2^{-1}\right)}\left((1+\kappa\sigma) - e_1\left((1+\kappa\sigma)^2+\beta\kappa\sigma\right)\right) + (\beta e_2)^{T-t}(1+\kappa\sigma)\right](\bar{\pi}^*+\underline{r})
\end{aligned}$$

or

$$\pi_t = \begin{cases} -\underline{r} + c_1\,(\beta e_1)^{T-t} + c_2\,(\beta e_2)^{T-t} & \text{for } t = 0, 1, \ldots T \\ \bar{\pi}^* & \text{for } t > T \end{cases}$$

where

$$\begin{aligned}
c_1 &= \left(1-e_1e_2^{-1}\right)^{-1}\left((1+\kappa\sigma) - e_1\left((1+\kappa\sigma)^2+\beta\kappa\sigma\right)\right)(\bar{\pi}^*+\underline{r}) \\
c_2 &= \left((1+\kappa\sigma) - \frac{1}{1-e_1e_2^{-1}}\left((1+\kappa\sigma) - e_1\left((1+\kappa\sigma)^2+\beta\kappa\sigma\right)\right)\right)(\bar{\pi}^*+\underline{r}) \\
&= \left(1-e_1e_2^{-1}\right)^{-1}\left(\left(1+\kappa\sigma-e_2^{-1}\right)(1+\kappa\sigma)+\beta\kappa\sigma\right)e_1\,(\bar{\pi}^*+\underline{r}).
\end{aligned}$$

Given that $\underline{r} + \bar{\pi}^* < 0$, we have $c_2 < 0$. In addition, $c_1 + c_2 = (1+\kappa\sigma)(\bar{\pi}^*+\underline{r}) < 0$, so that in period T, $\pi_T = -\underline{r} + c_1 + c_2 < -\underline{r}$.

To solve for output, we iterate forward (55) to obtain

$$\begin{aligned}
\hat{Y}_t &= E_t\sum_{j=1}^{T-t}\sigma(\pi_{t+j}+\underline{r}) + E_t\hat{Y}_T = \sum_{j=1}^{T-t}\sigma(c_1\,(\beta e_1)^{T-t-j} + c_2\,(\beta e_2)^{T-t-j}) + \bar{Y} + \sigma(\bar{\pi}^*+\underline{r}) \\
&= \bar{Y} + \sigma\left[\bar{\pi}^* + \underline{r} + c_1\frac{1-(\beta e_1)^{T-t}}{1-\beta e_1} + c_2\frac{1-(\beta e_2)^{T-t}}{1-\beta e_2}\right] \\
&= \bar{Y} + \sigma\left[\bar{\pi}^* + \underline{r} + \frac{c_1}{1-\beta e_1} + \frac{c_2}{1-\beta e_2} - c_1\frac{(\beta e_1)^{T-t}}{1-\beta e_1} - c_2\frac{(\beta e_2)^{T-t}}{1-\beta e_2}\right] \\
&= c_0 - \sigma c_1\frac{(\beta e_1)^{T-t}}{1-\beta e_1} - \sigma c_2\frac{(\beta e_2)^{T-t}}{1-\beta e_2},
\end{aligned}$$

where $c_0 \equiv \bar{Y} + \sigma\left(\bar{\pi}^* + \underline{r} + \frac{c_1}{1-\beta e_1} + \frac{c_2}{1-\beta e_2}\right)$, and the second equality is obtained by noting that (55) implies $\hat{Y}_T = \bar{Y} + \sigma(\bar{\pi}^*+\underline{r})$.