

Optimal Time-Consistent Monetary Policy in the New Keynesian Model with Repeated Simultaneous Play

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Abstract

We solve for the optimal time-consistent monetary policy in the New Keynesian model with repeated simultaneous play between the monetary authority, households, and firms. Recent work on optimal time-consistent monetary policy has emphasized the existence of multiple Markov perfect equilibria in the New Keynesian model (e.g., King and Wolman, 2004). In this paper, we show that this multiplicity is not intrinsic to the New Keynesian model itself, but is instead driven by an auxiliary timing assumption by previous authors that play is “repeated Stackelberg”—in which the monetary authority must pre-commit each period to a value for the monetary instrument—as opposed to repeated simultaneous, in which the monetary authority and the private sector determine the economic equilibrium simultaneously and jointly each period. A contribution of our paper is to show how to define the game between the monetary authority, households, and firms with repeated simultaneous play and aggregate resource constraints. We show that the repeated simultaneous play assumption is the proper generalization of the large existing literature on linear-quadratic optimal monetary policy under uncertainty (e.g., Woodford, 2003, Svensson and Woodford, 2003, 2004). We highlight and discuss additional advantages of the repeated simultaneous play assumption. Finally, we derive a closed-form solution for the set of all possible Markov perfect equilibria in the two-period Taylor contracting version of the New Keynesian model with simultaneous play and show that the equilibrium in that model is unique.

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1 Introduction

Many countries have witnessed periods of high and persistent inflation, as the U.S. did in the 1970s. What caused these episodes? More importantly, how can central banks or governments prevent them from recurring in the future? These are questions that remain important topics of research in monetary economics.

One prominent explanation, due originally to Barro and Gordon (1983) and refined by Chari, Christiano, and Eichenbaum (1998), is that the time-consistency problem of monetary policy in the absence of a commitment mechanism (Kydlund and Prescott, 1977) leads to a multiplicity of possible equilibria, some with substantially higher inflation rates than others. These theories suffer from two important drawbacks, however: First, they have little empirical content, because the enormous number and range of equilibria they allow make the theories essentially impossible to reject or critique on the basis of observation. Second, many of the equilibria implied by the models are complex and require a great deal of sophistication and coordination across a large number of atomistic agents in order to arise.

In response to these shortcomings, the literature has turned to the much simpler and smaller class of *Markov perfect equilibria*, in which agents may only condition their actions on economic fundamentals—i.e., the state variables of the model. A striking finding of this literature (e.g., Albanesi, Chari, and Christiano, 2003, King and Wolman, 2004) is that there exist multiple Markov perfect equilibria in standard, New Keynesian dynamic general equilibrium models. An implicit conclusion of these studies is that the U.S. (and other countries) could once again find itself caught in a bad “expectations trap” for inflation and a possible repeat of the 1970s. A more explicit conclusion of these studies is that linear-quadratic approximations to the New Keynesian model that have been used by many authors to compute optimal time-consistent monetary policy (e.g., Clarida, Gali, and Gertler, 1999, Svensson and Woodford, 2003, Woodford, 2003) are completely missing the most important features of the model.

In this paper, we show that multiple Markov perfect equilibria are not intrinsic to the New Keynesian model itself, but instead are driven by an auxiliary assumption about the timing of play in the model that previous authors have made for the sake of simplicity. In particular, a common feature across all of Albanesi, Chari, and Christiano (2003), Khan, King, and Wolman (2001), Dedola (2002), King and Wolman (2004), Siu (2005), and Armenter (2005) is a timing assumption in which each period is divided into two halves, with the monetary authority setting the policy instrument (an interest rate or the money supply) in the first half of the period, and the private sector responding in the second half of the period, in a repeated Stackelberg fashion. We relax this assumption and instead allow the monetary authority to play simultaneously with the private sector each period. We show that the optimal monetary policy in the New Keynesian model with two-period Taylor contracts leads to a unique equilibrium under repeated simultaneous play.

One of the primary contributions of our paper is to show how to think about and model repeated simultaneous play in a formal macroeconomic dynamic game. In standard games of industry competition, firms

may set whatever prices or produce whatever quantities they desire, subject only to their own technological constraints, so simultaneous play poses no particular problems. In a macroeconomic model, by contrast, the game comprises the *entire economy*, so that if a positive measure of firms or households were to deviate from equilibrium play—or if a large player, such as the central bank, were to deviate—then the economy’s aggregate resource constraints would be violated. For example, even though any individual household is unrestricted in its choice of labor and consumption, the set of all households collectively *must* supply enough labor to produce the amount of output that is collectively consumed. Similarly, the central bank cannot supply more money than households collectively demand. How can we define the game so that non-violation of the economy’s aggregate resource constraints is assured? The issue is more than academic: as Bassetto (2002, 2005) shows in analyzing the fiscal theory of the price level, a game that does not explicitly respect the economy’s aggregate resource constraints under all possible play—including out-of-equilibrium play—is not well-specified and can lead to pseudo-equilibria that are in fact completely invalid when the game is specified more carefully. We show how to formally model the standard New Keynesian model as a dynamic game in which the economy’s aggregate resource constraints are respected.

We are not the first authors to find repeated simultaneous play to be more appealing than repeated Stackelberg play for the study of optimal monetary policy. Indeed, the large literature on optimal time-consistent policy in linear-quadratic models dating back to the 1980s works largely within this framework. The assumption is made most explicitly in papers which allow for stochastic shocks and imperfect observation by policymakers and the private sector of the true state of the economy, such as Pearlman, Currie, and Levine (1986), Pearlman (1992), Svensson and Woodford (2003), and Woodford (2003). This is because, within the imperfect information framework, the optimal policy cannot be expressed simply as a function of the (unobserved) predetermined variables of the model, but instead is a function of the policymaker’s and the private sector’s optimal estimate of those variables conditional on all information at date t , including output and inflation. This simultaneity (or “circularity” in the terminology of Svensson and Woodford) causes the policymaker’s optimal choice of instrument to depend upon the private sector’s choices for output and inflation which in turn depend upon the policymaker’s instrument choice. One of the contributions of our paper is to show how to think about and model this simultaneity within the framework of a formal dynamic game.

We also argue that the assumption of repeated simultaneous play by the monetary authority is more appealing than that of repeated Stackelberg play, for a number of reasons. First, when the central bank and the private sector play simultaneously, it no longer matters whether the central bank’s instrument is the money stock or the nominal interest rate—the set of equilibria are identical under either assumption—an equivalence which does not hold under repeated Stackelberg play, as shown in section 5, below, and in Dotsey and Hornstein (2006). We regard this equivalence between monetary instruments as an appealing feature

of our timing assumption because most central banks in practice adjust the money supply to maintain a short-term interest rate target, and it is not at all clear which should be regarded as the monetary policy instrument if the two are not equivalent. Second, central banks around the world continuously monitor and maintain a target for a short-term interest rate (or monetary aggregate) and are free to continuously adjust this target should unforeseen economic developments arise over the course of the month or quarter; thus, the idea of central banks “pre-committing” to a given level of the money supply or an interest rate is arguably at odds with the data.¹ Although one can address this second criticism by shrinking the length of a period in the Stackelberg model down to one day or even one hour, it is still not clear why one would want to treat the central bank and the private sector so asymmetrically as to have one or the other always play first. Moreover, shrinking the length of a period does not address the first criticism above. Third, much existing analysis of optimal time-consistent monetary policy in the New Keynesian model has been done using a linear-quadratic approximation within a simultaneous timing framework (e.g., Clarida, Gali, and Gertler, 1999, Svensson and Woodford, 2003, and Woodford, 2003), exactly the timing assumption that we argue should be used in general. Thus, our analysis—and not that in King and Wolman (2004), Albanesi, Chari, and Christiano (2003), and the other papers cited above—provides the proper benchmark with which to make judgments regarding the accuracy or possible misspecification of the LQ approach to optimal monetary policy.

An additional contribution of our paper relative to the previous literature is the computation of closed-form solutions for all possible Markov perfect equilibria in the two-period Taylor contracting version of the New Keynesian model. As a result, we are able to rigorously prove the existence and nonexistence of multiple equilibria in that model.

The remainder of the paper proceeds as follows. Section 2 defines the New Keynesian model of the private sector economy and the necessary conditions for a private sector equilibrium in that economy given an exogenous interest rate path. Section 3 presents the optimal policy problem for the central bank, formally defines the game being played, and defines the necessary conditions for a Markov perfect equilibrium in the game. Section 4 derives the closed-form solution to the model and proves the uniqueness of the equilibrium. Section 5 discusses the differences between simultaneous and Stackelberg timing assumptions in the simplest possible case of a model with only two periods. Section 6 concludes. A Technical Appendix contains details of the proofs in Section 4.

¹Although we rarely observe central banks changing their instrument between regularly scheduled meetings along the equilibrium path, this should not be interpreted as a structural constraint on their feasible set of policies or their out-of-equilibrium behavior.

2 The Private Sector Economy and Private Sector Equilibrium

We focus in this paper on a standard dynamic New Keynesian model of the economy with monopolistically competitive intermediate goods markets, two-period Taylor price contracts, perfectly competitive factor markets, flexible wages, and homogeneous labor. For tractability and simplicity, we abstract away from endogenous variation in the capital stock.

Before turning to the question of optimal monetary policy, it will be useful to first define the New Keynesian economy and the corresponding game Γ_0 for the case of an *exogenous* interest rate process $\{r_t\}$. This will allow us to address a number of key issues without the additional complications of having the central bank as an additional player in the game. We will defer until the next section the game Γ_1 that we are primarily interested in, in which the short-term nominal interest rate is set by an optimizing central bank.

2.1 Players in the Game Γ_0

2.1.1 Firms

Time is discrete and continues forever. There is a continuum of atomistic firms in the economy indexed by $i \in [0, 1)$. The measure of firms is constant over time. Each firm is a player in the game Γ_0 . At each time t , each firm produces a single, differentiated product, also indexed by i , according to the linear production function:

$$y_t(i) = l_t(i),$$

where $y_t(i)$ is the quantity of output produced and $l_t(i)$ the quantity of labor employed by firm i in period t .²

The price of each good i must be set for two periods in a staggered Taylor fashion, with firms in $[0, 1/2)$ free to change their price in even periods and firms in $[1/2, 1)$ free to change their price in odd periods. Each firm i must satisfy demand for its product in every period at its posted price $p_t(i)$, hiring whatever labor inputs are necessary. Nominal profits for firm i in period t are given by:

$$\Pi_t(i) \equiv p_t(i)y_t(i) - w_t l_t(i),$$

where w_t is the nominal wage in period t . Firms are owned by households, below, and distribute all profits or losses to all households equally every period. We define aggregate firm profits by:

$$\Pi_t \equiv \int_0^1 \Pi_t(i) di. \tag{1}$$

²There is no loss of generality in assuming linearity (as opposed to homotheticity of lower degree) because households' disutility of working, defined below, will be homothetic with arbitrary parameter χ .

2.1.2 Households

The economy also contains a continuum of atomistic households indexed by $j \in [0, 1]$. Each household is a player in the game Γ_0 . There is no population growth. At each date t , each household j receives a utility flow (payoff) according to:

$$\frac{C_t(j)^{1-\varphi} - 1}{1-\varphi} - \chi_0 \frac{L_t(j)^{1+\chi}}{1+\chi}, \quad (2)$$

where $C_t(j)$ is the quantity of the final good consumed and $L_t(j)$ the quantity of labor supplied by household j in period t . The household discounts future utility flows (payoffs) at the rate β per period. Households can buy and sell risk-free one-period nominal bonds which pay interest rate r_t . The household faces an intertemporal budget constraint defined by the asset accumulation equation:

$$B_t(j) = (1 + r_{t-1})B_{t-1}(j) + w_t L_t(j) + \Pi_t - P_t C_t(j), \quad (3)$$

where Π_t denotes the household's aliquot share of aggregate firm profits, w_t denotes the nominal wage, P_t the price of the consumption good, and $B_t(j)$ the household's stock of one-period bonds at the end of period t , which we require to satisfy the transversality condition:

$$\lim_{T \rightarrow \infty} \tilde{E}_{jt} B_T(j) \prod_{s=t}^{T-1} (1 + r_s)^{-1} \geq 0, \quad (4)$$

where \tilde{E}_{jt} denotes the mathematical expectation conditional on household j 's information set at date t (specified below).

2.2 Two Mechanisms (not Players) in the Game Γ_0

The economy also consists of two mechanisms—which are *not* players in the game Γ_0 —that are not formally necessary but help to clarify the exposition of the model and the game. For example, one can define a continuum of competitive final good-producing firms as players in the game Γ_0 , but then those additional players (which are uninteresting) must be carried through the formal definition of the game.³ Rather than distract attention from the essential features of the game, we simply assume that the aggregation of intermediate goods into final goods happens automatically and non-strategically, as specified below.

³Alternatively, one can drop the final good entirely and define the household's utility function directly in terms of the individual goods i rather than an aggregate consumption good, but then the action space of each household must be defined as a function space over all the individual goods, which introduces additional and uninteresting complications to the problem.

2.2.1 Goods Aggregator

The economy has an automatic *intermediate good aggregator*—that is *not* a player in the game Γ_0 —that transforms intermediate goods i into the final consumption good according to:

$$Y_t \equiv \left[\int_0^1 y_t(i)^{1/(1+\theta)} di \right]^{1+\theta}. \quad (5)$$

The transformation is perfectly competitive, i.e.,

$$y_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-(1+\theta)/\theta} Y_t, \quad (6)$$

where P_t is the Dixit-Stiglitz aggregate price index,

$$P_t \equiv \left[\int_0^1 p_t(i)^{-1/\theta} di \right]^{-\theta}. \quad (7)$$

2.2.2 Walrasian Auctioneer

The economy also has an automatic Walrasian auctioneer—that is *not* a player in the game Γ_0 —that ensures that all markets clear at each time t . As we will specify in detail below, a firm that is resetting price in period t does *not* set a price but rather submits a *price schedule* to the Walrasian auctioneer that is a function of relevant variables dated t , such as w_t , P_t , r_t , and Y_t . We defer the details and discussion of this assumption to the next section, but note here that this assumption on the firm's action space is necessary if we wish to allow the firm to set a price in the game that is the usual function of aggregate variables dated t . Households likewise submit joint consumption demand-labor supply schedules to the auctioneer as functions of relevant variables dated t , such as w_t , P_t , r_t , and Π_t . The auctioneer then determines the values of w_t , $p_t(i)$, $C_t(j)$, and $L_t(j)$ for all i and j that clear the final good market, the labor market, and the bond market at time t , i.e.:

$$\int_0^1 C_t(j) dj = Y_t, \quad (8)$$

$$\int_0^1 l_t(i) di = \int_0^1 L_t(j) dj, \quad (9)$$

and

$$\int_0^1 B_t(j) dj = 0. \quad (10)$$

Equation (8) states that aggregate consumption equals output of the final good, (9) that aggregate labor demand equals aggregate labor supply, and (10) that bonds are in zero net aggregate supply. We show below that such an equilibrium exists.

Why do we need the assumption of an auctioneer? First, the auctioneer makes explicit the mechanism by which the economy's aggregate resource constraints are enforced. In standard games of industry competition,

players may set whatever prices or produce whatever quantities they desire, subject only to their own technological constraints. Here, the game comprises the *entire economy*, and if a positive measure of firms or workers were to deviate from equilibrium play—or if a large player such as the monetary authority were to deviate—then the economy’s aggregate resource constraints (8)-(10) could be violated without the intervention of the Walrasian auctioneer.

The second reason to assume a Walrasian auctioneer is that it explicitly allows firms and households to play price *schedules* and consumption demand and labor supply *schedules*. Without this ability, firms and households cannot condition their play on the outcome of key variables dated t , such as wages w_t , prices P_t , and the interest rate r_t . For example, an auctioneer is clearly implicit in the literature on linear-quadratic optimal control with imperfect information (e.g., Pearlman et al., 1986, Pearlman, 1992, Svensson and Woodford, 2003, Woodford, 2003), where simultaneous play by the monetary authority and the private sector is explicitly assumed. Without the assumption that firms and households play entire schedules that are then cleared by an auctioneer, such explicitly simultaneous play is impossible. Our assumption of an auctioneer thus makes explicit a device which is implicitly at work in much of the earlier literature.

2.3 Information Sets in the Game Γ_0

Define the *publicly available information set* at time t , I_t , to be the set of all aggregate variables dated $t - 1$ and earlier:

$$I_t \equiv \{L_s, P_s, p_s, r_s, w_s, Y_s, \Pi_s : s < t\} \quad (11)$$

where in addition to the aggregate variables defined previously, we define aggregate labor:

$$L_t \equiv \int_0^1 L_t(j) dj, \quad (12)$$

and the average price set by firms who change price in period t :

$$p_t \equiv \left[\int_{\frac{t \bmod 2}{2}}^{\frac{t \bmod 2 + 1}{2}} p_t(i)^{-1/\theta} di \right]^{-\theta}. \quad (13)$$

Note that we do not define aggregate bond holdings as a variable, because the Walrasian auctioneer will constrain these to be in zero net aggregate supply every period. Similarly, we do not define aggregate consumption because the auctioneer will constrain it to equal aggregate output in every period. Also note that the time- t realizations of variables are *not* in the information set at time t , because they depend on the actions that are chosen by firms and households at time t .

Each firm i and household j at time t observes the set I_t and also the history of its own actions and outcomes in all previous periods. Thus, we define the information set of firm i at time t by:

$$I_{it} \equiv I_t \cup \{p_s(i), y_s(i), \Pi_s(i) : s < t\}, \quad (14)$$

and the information set of household j at time t by:

$$I_{jt} \equiv I_t \cup \{C_s(j), L_s(j), B_s(j) : s < t\}. \quad (15)$$

Note that this implies that individual firms and households are *anonymous* in the sense that the histories of their individual actions and outcomes are unobserved by the other players of the game.

We will use \tilde{E}_t to denote the mathematical expectation operator conditional on information set I_t , \tilde{E}_{it} to denote the mathematical expectation conditional on information set I_{it} , and \tilde{E}_{jt} to denote the mathematical expectation conditional on information set I_{jt} . We state the following observation as a proposition:

Proposition 1 *For $X \in \{L, P, p, r, w, Y, \Pi\}$, the following equalities hold for all i, j , and t : $\tilde{E}_{it}X_t = \tilde{E}_tX_t$, $\tilde{E}_{it}X_{t+1} = \tilde{E}_tX_{t+1}$, $\tilde{E}_{jt}X_t = \tilde{E}_tX_t$, and $\tilde{E}_{jt}X_{t+1} = \tilde{E}_tX_{t+1}$.*

Proof. The proof follows from two observations. First, firm i is anonymous, so the actions of other firms and households cannot depend on $\{p_s(i), y_s(i), \Pi_s(i) : s < t\}$, by assumption. Second, firm i has measure zero, so changes in $\{p_s(i), y_s(i), \Pi_s(i) : s < t\}$ have no direct effect on X_t or X_{t+1} , holding $\{p_s(k), y_s(k), \Pi_s(k) : s < t\}$, $k \neq i$, constant. It follows that X_t and X_{t+1} are independent of $\{p_s(i), y_s(i), \Pi_s(i) : s < t\}$.

Similarly, household j is anonymous and has measure zero, so X_t and X_{t+1} are likewise independent of $\{C_s(j), L_s(j), B_s(j) : s < t\}$. ■

2.4 Action Spaces in the Game Γ_0

If $i \in [0, 1/2)$, then in every even period t firm i submits a price *schedule*, or function, to the Walrasian auctioneer, which can depend on the aggregate variables of the model that are realized at date t : $L_t, P_t, p_t, r_t, w_t, Y_t$, and Π_t . Thus, the action space of each firm is a *function*, rather than a real number. We restrict the firm to submit price schedules that are continuous functions of these variables, although in equilibrium this restriction is not binding. The action space of each firm i in each period t is thus $C(\mathbb{R}^7)$, the space of continuous functions from \mathbb{R}^7 into \mathbb{R}_+ . In odd periods, firm i has empty action space.

If $i \in [1/2, 1)$, then the action spaces of firm i are the same as above, except that the firm submits a price schedule to the auctioneer in odd periods, and has empty action space in even periods.

For household j in period t , the household likewise submits a joint consumption demand-labor supply schedule to the auctioneer that depends on aggregate variables realized in period t : $L_t, P_t, p_t, r_t, w_t, Y_t$, and Π_t . Again, we restrict the household to play functions that are continuous, although in equilibrium this restriction is not binding. The action space of each household j in each period t is thus $C(\mathbb{R}^7, \mathbb{R}_+^2)$, the space of continuous functions from \mathbb{R}^7 into \mathbb{R}_+^2 .

Note that firms and households all play simultaneously in each period t , although the simultaneity is different than in standard games of industry Bertrand or Cournot competition. The reason for the difference

is that in our model, the players of the game must collectively obey the aggregate resource constraints. This is impossible to guarantee unless each household and firm plays a function rather than a real number, where the functions explicitly depend on prices, wages, and aggregate demand conditions in date t . The auctioneer then determines the equilibrium that satisfies the resource constraints.

It is also worth emphasizing the distinction between the action spaces and *strategies* of players in the game. For example, denote household j 's action space above by $A(j)$, and the history $\{L_s, P_s, p_s, r_s, w_s, Y_s, \Pi_s, C_s(j), L_s(j), B_s(j) : s < t\} \in \mathbb{R}^\omega$ by $h^t(j)$. Then a *strategy* for household j is a sequence of functions $\{\sigma_t(j)\}$, $t \in \mathbb{Z}$, $\sigma_t(j) : \mathbb{R}^\omega \rightarrow A(j)$, that specify what action household j will play at each time t after observing history $h^t(j)$. Just because the household's *action space* includes functions of variables dated t , rather than real numbers, does not mean that the household is constrained in any way from playing *strategies* that are functions of the entire history $h^t(j)$ of observed outcomes. In fact, it is well known from dynamic programming that the household's optimal strategy is to play schedules for $C_t(j)$ and $L_t(j)$ that depend on the household's inherited stock of bonds $B_{t-1}(j)$, as well as any other state variables that may influence the future stochastic behavior of $\{P_s, r_s, w_s, \Pi_s\}$, $s > t$.

2.5 Optimality Conditions in the Game Γ_0

2.5.1 Firm Optimality Conditions

The firm's objective (payoff) in every other period t is to maximize the expected present discounted value of profits over the two periods in which that price remains in effect:

$$\tilde{E}_{it} [\Pi_t(i) + Q_{t,t+1} \Pi_{t+1}(i)],$$

taking the demand curve for the firm's product (6) as given and where $Q_{t,t+1}$ denotes the stochastic discount factor by which the economy values random nominal income at date $t+1$ in monetary units at date t .⁴

Firm optimization implies that, for a firm that is permitted to reset its price in period t , the optimal price $p_t^*(i)$ satisfies the first-order condition:

$$\frac{\partial}{\partial p_t(i)} \tilde{E}_{it} [\Pi_t(i) + Q_{t,t+1} \Pi_{t+1}(i)] = 0,$$

Evaluating this derivative yields the optimality condition:

$$p_t^*(i) = (1 + \theta) \frac{\tilde{E}_t P_t^{(1+\theta)/\theta} Y_t w_t + \tilde{E}_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{\tilde{E}_t P_t^{(1+\theta)/\theta} Y_t + \tilde{E}_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}} \quad (16)$$

where we have used Proposition 1 to replace the \tilde{E}_{it} operator with \tilde{E}_t in (16).

⁴Such a stochastic discount factor exists—see, e.g., Cochrane (2001) and the references therein. In equilibrium, all households will be identical and thus the stochastic discount factor will be the marginal rate of substitution of the representative household.

An optimal price $p_t^*(i)$ must also satisfy the second-order condition:

$$\frac{\partial^2}{\partial p_t(i)^2} \tilde{E}_{it} [\Pi_t(i) + Q_{t,t+1} \Pi_{t+1}(i)] \leq 0,$$

and ex ante nonnegative expected profit (non-shut-down) condition:

$$\tilde{E}_{it} [\Pi_t(i) + Q_{t,t+1} \Pi_{t+1}(i)] \geq 0,$$

which yield, respectively:

$$p_t^*(i) \leq (1 + 2\theta) \frac{\tilde{E}_t P_t^{(1+\theta)/\theta} Y_t w_t + \beta \tilde{E}_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{\tilde{E}_t P_t^{(1+\theta)/\theta} Y_t + \beta \tilde{E}_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}},$$

and

$$p_t^*(i) \geq \frac{\tilde{E}_t P_t^{(1+\theta)/\theta} Y_t w_t + \beta \tilde{E}_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{\tilde{E}_t P_t^{(1+\theta)/\theta} Y_t + \beta \tilde{E}_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}}.$$

Note that, given (16), both inequalities above are satisfied automatically so long as the markup $\theta > 0$, which we have assumed.

Given (16), what price schedule should the firm play at time t ? Even though the time- t realizations of variables such as P_t , r_t , w_t , and Y_t are not in the firm's information set at time t , the firm is still able to submit a schedule for $p_t(i)$ to the Walrasian auctioneer that is a function of those time- t realizations, by our definition of the firm's action space. Thus, the optimal action by firm i at time t is the price schedule:

$$p_t^*(i) = (1 + \theta) \frac{P_t^{(1+\theta)/\theta} Y_t w_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{P_t^{(1+\theta)/\theta} Y_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}} \quad (17)$$

which differs from (16) in that the conditioning of $p_t^*(i)$ on P_t , w_t , and Y_t is explicit (the \tilde{E}_t operator on those terms is dropped) and where we define E_t to be the mathematical expectation conditional on information set $I_t \cup \{L_t, P_t, p_t, r_t, w_t, Y_t, \Pi_t\}$. Equation (17) is just the standard first-order condition from the literature (e.g., Erceg et al., 2000, Woodford, 2003), which we have now derived rigorously within the framework of the game Γ_0 .

We state one final observation as a proposition:

Proposition 2 *The optimal choice of price schedule $p_t^*(i)$ is the same for all firms i that reset price in period t . We denote this optimal price schedule, given by (17), by p_t^* .*

Proof. The right-hand side of optimality condition (17) is identical for all firms i . ■

Note that we do not need to assume symmetry because the actions of other firms do not enter into (17)—only aggregate variables enter into the equation. Thus, symmetry of the optimal price across firms is an implication of the model and not an assumption.

2.5.2 Household Optimality Conditions

Household j at each time t faces a standard dynamic programming problem: choose $C_t(j)$ and $L_t(j)$ (and state-contingent plans for the future values of these variables) to maximize expected welfare (2) subject to the constraints (3)–(4), taking initial bond holdings $B_{t-1}(j)$ and the stochastic process for $\{L_s, P_s, p_s, r_s, w_s, Y_s, \Pi_s\}$, $s \geq t$, as given. As was the case for firms, the time- t realizations of the variables $L_t, P_t, p_t, r_t, w_t, Y_t$, and Π_t are not yet in the household's information set at time t , but the household is permitted to submit schedules for $C_t(j)$ and $L_t(j)$ to the Walrasian auctioneer that are functions of those time- t realizations.

The solution to this programming problem for the optimal functions $C_t^*(j)$ and $L_t^*(j)$ is well known, though closed-form solutions do not exist in general. Optimal consumption behavior satisfies the Euler equation:

$$C_t^*(j)^{-\varphi} = \tilde{E}_{jt} \beta (1 + r_t) \frac{P_t}{P_{t+1}} C_{t+1}^*(j)^{-\varphi}, \quad (18)$$

optimal labor supply sets the intratemporal marginal rate of substitution equal to the real wage:

$$\chi_0 L_t^*(j)^{\chi} = \tilde{E}_{jt} \frac{w_t}{P_t} C_t^*(j)^{-\varphi}, \quad (19)$$

and the transversality condition (4) is satisfied with equality:

$$\tilde{E}_{jt} \sum_{T=t}^{\infty} R_{t,T} P_T C_T^*(j) = (1 + r_{t-1}) B_{t-1}(j) + \tilde{E}_{jt} \sum_{T=t}^{\infty} R_{t,T} [w_T L_T^*(j) + \Pi_T], \quad (20)$$

where $R_{t,T} \equiv \prod_{s=t}^{T-1} (1 + r_s)^{-1}$.

Household j 's optimal choice of consumption demand-labor supply schedule at time t , $\{C_t^*(j), L_t^*(j)\}$, is implicitly defined by the three well-known equations (18)–(20). We state the following observation as a proposition:

Proposition 3 *Suppose that $B_{t-1}(j) = B_{t-1}(k)$ for two households j and k . Then the optimal actions $\{C_t^*(j), L_t^*(j)\} = \{C_t^*(k), L_t^*(k)\}$. If $B_{t-1}(j)$ is the same for all households $j \in [0, 1]$ except possibly a set S of measure zero, then the optimal actions $\{C_t^*(j), L_t^*(j)\}$ are the same for every household $j \notin S$, and we denote these actions by $\{C_t^*, L_t^*\}$.*

Proof. As was the case for firms, the household's optimal actions at time t , implicitly defined by (18)–(20), depend on variables dated t , such as r_t, w_t, P_t , and Π_t . Similarly, because the household can play functions that depend on variables dated t , it is effectively free to condition its expectation operator \tilde{E}_{jt} in (18)–(20) on these variables as well, so that the operator \tilde{E}_{jt} can be thought of as conditional on all aggregate and household- j -specific variables dated t and earlier. Finally, note that the optimality conditions (18)–(20) for household j do not depend on the actions of other households except through the aggregate quantities P_t, w_t , etc. ■

Again, note that the symmetry across households is not an assumption but rather an equilibrium implication of the model.

The stochastic discount factor by which the economy values future nominal income at date $t + 1$ is then given, in equilibrium, by:

$$Q_{t,t+1}^* = \beta \frac{(C_{t+1}^*)^{-\varphi}}{(C_t^*)^{-\varphi}} \frac{P_t}{P_{t+1}}. \quad (21)$$

2.6 State Variables of the Game Γ_0

Contrary to conventional wisdom (e.g., Woodford, 2003, King and Wolman, 2004), the game Γ_0 has *three* sets of state variables rather than none. First, there is the continuum of household-specific bond holdings, $B_{t-1}(j)$, $j \in [0, 1]$, which are state variables of the individual households' dynamic programming problems. Second, there is the cross-sectional dispersion of prices:

$$\Delta_{t-1} \equiv \int_0^1 \left(\frac{p_{t-1}(i)}{p_{t-1}} \right)^{-1/(1+\theta)} di. \quad (22)$$

Third, there is the set of state variables that govern the stochastic process for $\{r_t\}$, such as r_{t-1} . Note that the inherited average price from last period, p_{t-1} , is not a state variable, since all the nominal quantities in the model can be normalized by p_{t-1} and not affect the players' action spaces or payoffs.

The conventional wisdom is correct, however, in the sense that the first two sets of state variables turn out to be either irrelevant or uninteresting for our analysis of the Markov perfect equilibria of the game. First, for the case of price dispersion Δ_{t-1} , a feature of two-period Taylor contracting is that Δ_t is always equal to 1 in equilibrium (as a corollary to Proposition 2), even if the inherited value of Δ_{t-1} were out of equilibrium and unequal to 1. Thus, even if players at time t inherit a value for $\Delta_{t-1} \neq 1$, all players will correctly expect it to return to zero from period t onward. Thus, while an out-of-equilibrium value of $\Delta_{t-1} \neq 1$ would affect outcomes in period t , it would not affect households' and firms' expectations of outcomes in period $t + 1$ and later. Thus, the state variable Δ_{t-1} poses no additional complications for our analysis and indeed will always be equal to 1 in equilibrium.

Second, we are not interested in distributional issues, so we will restrict attention at the outset to the case where the initial distribution of bond holdings across households is symmetric—i.e., $B_{t_0-1}(j) = 0$ for all $j \in [0, 1]$ except possibly a set of measure zero. In that case, all households with $B_{t_0-1}(j) = 0$ will choose the same $\{C_t^*(j), L_t^*(j)\} = \{C_t^*, L_t^*\}$ and the same $B_t^*(j) = B_t^* = 0$ in equilibrium for every time t . Thus, following Phelan and Stachetti (2001), we will leave unspecified the future behavior of firms and households when a positive measure of households deviate from equilibrium (i.e., we will assert that they will continue to play a Markov perfect equilibrium, but we will not specify precisely what that equilibrium behavior is). We will still be able to verify that no household has an incentive to deviate from any Markov perfect equilibrium path that we compute.

2.7 Private Sector Markov Perfect Equilibrium in the Game Γ_0

The game Γ_0 defined by the players, action spaces, information sets, and payoffs above has a potentially large number of equilibria (even without a central bank) if agents play trigger strategies based on past histories. However, given the huge number of players involved, it is difficult to see how all of them could coordinate on the same trigger strategy to achieve most of these equilibria. Since such extraordinary coordination seems implausible, we follow the literature and restrict attention to the much smaller and simpler class of Markov perfect equilibria, in which each agent chooses actions based only on his or her own payoff-relevant, or state, variables (e.g., Fudenberg and Tirole, 2001).

We define a Markov perfect “Private Sector Equilibrium” of the aggregate variables of the game Γ_0 at time t_0 as follows:

Definition 1 *Given a value Δ_{t_0-1} , values $B_{t_0-1}(j)$ for all $j \in [0, 1]$ with $B_{t_0-1}(j) = 0$ for almost all j , and an exogenous stochastic process for $\{r_t\}$ with state variables at each time $t \geq t_0$ given by the vector X_t , a Private Sector Equilibrium (PSE) at time t_0 is a stochastic process for $\{L_t, P_t, p_t, w_t, Y_t, \Pi_t\}$, $t \in \mathbb{Z}$, that (i) satisfies conditions (5)–(21) for all i , for all j , and for all $t \geq t_0$, and (ii) for all $t > t_0$, $\{L_t, P_t, p_t, w_t, Y_t, \Pi_t\}$ is independent of any variable Ξ_s , $s < t$, that is not an element of X_t .*

Condition (ii) follows from the Markovian restriction on firms’ and households strategies and the observation that $\Delta_t = 0$ and $B_t^*(j) = 0$ for almost all j for all $t \geq t_0$. It follows that the only state variables that remain relevant for the stochastic process $\{L_t, P_t, p_t, w_t, Y_t, \Pi_t\}$ are those that govern the exogenous stochastic process $\{r_t\}$.

Note that we have omitted the individual $y_t(i)$ and $p_t(i)$ from the definition of Private Sector Equilibrium above, as the equilibrium values of the $p_t(i)$ are completely determined by those of p_t^* and p_{t-1}^* , and the $y_t(i)$ are completely determined by equation (6). Thus, these sectoral variables are extraneous as far as determining the equilibrium values of the aggregate variables of the model.

[Maybe we should check/prove uniqueness of MPE for the game Γ_0 , given a set of assumptions about the process $\{r_t\}$.]

3 The Optimal Policy Problem

We now turn to the question of the optimal time-consistent policy for $\{r_t\}$ in the above model. We thus wish to add one additional player to the game—an optimizing central bank—and denote this new game by Γ_1 . We then calculate the optimal strategies of the central bank and the set of possible Markov perfect equilibria.

3.1 The Central Bank and the Game Γ_1

The central bank differs from households and firms in two key respects. First, the central bank is a large player while households and firms are atomistic—thus, while households and firms take aggregate quantities as given, the central bank understands that its choice of nominal interest rate r_t changes the consumption and labor choices of households (through the optimal strategy conditions (18)–(20)) and the prices set by firms (through the optimal strategy condition (17)). It is this strategic interaction between interest rate setting and aggregate conditions that is at the heart of central banking.

The second difference is that the central bank faces a dynamic inconsistency between its current plans and its future policy choices. The reason is that the private sector’s expectation about future policy has an effect on current economic outcomes, giving the central bank an incentive to promise a future course for policy today that it may wish to renege on tomorrow.⁵ This makes the problem of deriving optimality conditions much more difficult for the case of the central bank than it was for households and firms, and requires us to put considerably more structure on the permissible strategy functions of the central bank than was necessary for households and firms.

In each period t , the central bank sets a value for the nominal one-period interest rate r_t . We show in section 5, below, that defining the monetary instrument to be the money supply (and appending a private-sector money demand function to the model) has no effect on our results—indeed, we regard this equivalence as an appealing feature of our assumption of simultaneous play between the central bank and the private sector.

The central bank’s payoff each period is the average welfare across all households:

$$\int_0^1 \left[\frac{C_T(j)^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_T(j)^{1+\chi}}{1+\chi} \right] dj. \quad (23)$$

which the central bank discounts at the rate β per period.

The central bank at time t chooses an optimal value for the interest rate r_t given the households’ choices for consumption demand-labor supply schedules $\{C_t^*, L_t^*\}$ and firms’ choices for price schedules p_t^* . The restrictions we place on the central bank’s behavior are, first, that the central bank has no ability to commit to future promises for its policy instrument, so that it operates under “discretion”, and second, that its policy choice for r_t at time t cannot depend on any variable dated $t - 1$ or earlier unless that variable is a fundamental “state” variable of the model, so that we restrict attention to Markov perfect equilibria.

⁵In contrast, the private sector faces no such issues in our model: the budget and resource constraints that they face are independent of expectations, so that their problems can be solved by classic dynamic programming.

3.2 Motivation for Simultaneous Play in the Game Γ_1

We assume here that households, firms, and the central bank all play *simultaneously* in each period t in the game Γ_1 . By contrast, the previous literature has made the assumption that households and firms play simultaneously in each period t , but that the central bank must precommit to a value for its monetary policy instrument (either a money stock or an interest rate) at the beginning of each period, with firms and households choosing prices, labor, and consumption afterward in a repeated Stackelberg fashion.

We find the assumption of simultaneous play between the central bank and the private sector to be the most natural one for a number of reasons. First, under simultaneous play, it makes no difference whether one defines the monetary instrument to be the short-term nominal interest rate or the money supply—the set of possible equilibria under either assumption for the monetary policy instrument is exactly the same (as we show in section 5). In contrast, this equivalence across monetary instruments does *not* hold under the repeated Stackelberg timing assumption, as shown by Dotsey and Hornstein (2006) and as we show in section 5, below.⁶ We regard this equivalence between monetary instruments as appealing because most central banks in practice adjust the money supply to maintain a short-term interest rate target, and it is not at all clear which should be regarded as the monetary policy instrument if the two are not equivalent.

Second, central banks continuously monitor economic conditions and have the freedom and ability to change the monetary instrument continuously, as needed. Thus, the assumption that central banks must precommit to a fixed value of the monetary instrument is at odds with the data—although most central banks find that, along the equilibrium path, they only rarely need to change the instrument between regularly scheduled meetings, this should not be interpreted as a structural constraint on their feasible set of policies or out-of-equilibrium behavior, should they be faced with a sudden deterioration in economic prospects. Although one can address this particular criticism by shrinking the length of a period in the Stackelberg model down to one day or even one hour, it is still not clear why one would want to treat the central bank and the private sector so asymmetrically as to have one or the other always play first (and shrinking the length of a period does not address the first criticism above).

Third, much existing analysis of optimal time-consistent monetary policy in the New Keynesian model has been done using a linear-quadratic approximation within a simultaneous timing framework (e.g., Clarida, Gali, and Gertler, 1999, Svensson and Woodford, 2003, and Woodford, 2003), exactly the timing assumption that we argue should be used in general. Thus, our timing assumption provides the proper benchmark with which to make judgments regarding the accuracy or possible misspecification of the LQ approach to optimal

⁶Intuitively, adding an additional equation for money demand and changing the monetary authority's instrument to the quantity of money is only guaranteed to yield the same set of equilibrium conditions if the quantity of money and the interest rate are determined simultaneously. Under Stackelberg play, the equivalence between the two instruments is broken because the private sector chooses the interest rate (and output, and expectations, etc.) *after* the monetary authority has already precommitted to a level for the money supply. We will return to this issue in more detail in section 6.

monetary policy.

Fourth and finally, the previous literature has typically found multiple equilibria under the optimal monetary policy with discretion. However, these authors' assumption that the monetary authority must precommit to a value for the monetary instrument—and cannot formulate its best response function in terms of the realization of variables at time t —is a substantial constraint on the central bank's ability to control the economy. It is an interesting question, then, whether restoring this control to the central bank (to an extent that is arguably more consistent with the data) would still admit the possibility of multiple equilibria.

3.3 State Variables of the Game Γ_1

Aggregate resource constraint in the labor market:

$$\frac{1}{2} \left(\frac{p_t^*}{P_t} \right)^{-(1+\theta)/\theta} Y_t + \frac{1}{2} \left(\frac{p_{t-1}^*}{P_t} \right)^{-(1+\theta)/\theta} Y_t = L_t. \quad (24)$$

Because state variables are of central importance to our definition of equilibrium, it is useful at this point to rewrite the optimality conditions of households and firms and the aggregate resource constraints in a way that makes it more transparent what the “state” of the economy is. We begin by defining the variable:

$$x_t \equiv \frac{p_t^*}{p_{t-1}^*},$$

which expresses firms' optimal price p_t^* today relative to their optimal price last period.

The aggregation equations (??) and (12) then simplify, respectively, to:

$$\frac{p_t^*}{P_t} = 2^{-\theta} (1 + x_t^{1/\theta})^\theta,$$

and

$$\frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}}. \quad (25)$$

Firms' optimal pricing condition (??) can be rewritten as:

$$2^{-\theta} (1 + x_t^{1/\theta})^\theta = \frac{1 + \theta}{1 - \tau} \frac{\chi_0 [Y_t L_t^\chi + \beta (1 + x_t^{1/\theta})^{1+\theta} h_{1t}]}{Y_t^{1-\varphi} + \beta (1 + x_t^{1/\theta}) h_{2t}}, \quad (26)$$

and households' optimal consumption choice (??) as:

$$Y_t^{-\varphi} (1 + x_t^{1/\theta}) = \beta (1 + r_t) h_{3t}, \quad (27)$$

where h_{1t} , h_{2t} , and h_{3t} denote the expectations:

$$h_{1t} \equiv E_t \frac{Y_{t+1} L_{t+1}^\chi}{(1 + x_{t+1}^{-1/\theta})^{1+\theta}}, \quad (28)$$

$$h_{2t} \equiv E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{-1/\theta}}, \quad (29)$$

$$h_{3t} \equiv E_t Y_{t+1}^{-\varphi} (1 + x_{t+1}^{-1/\theta}). \quad (30)$$

Note that equations (25)–(30) are sufficient to completely determine a private sector equilibrium of the model, up to a nominal scale variable p_{t-1}^* (or a choice of numeraire if we normalize $p_{t-1}^* = 1$). We state this as a proposition:

Proposition 4 *Necessary and sufficient conditions for a Private Sector Equilibrium for $\{L_t, x_t, Y_t, h_{1t}, h_{2t}, h_{3t}\}$ at time t_0 are that, for all $t \geq t_0$: (i) (L_t, x_t, Y_t) satisfy conditions (25)–(27), taking the expectations (h_{1t}, h_{2t}, h_{3t}) as given, and (ii) expectations are rational, so that (h_{1t}, h_{2t}, h_{3t}) are given by (28)–(30).*

Moreover, equations (25)–(30) rewrite the optimality conditions of firms and households completely in terms of variables that are determined at time t or later. Thus, the model has no state variable—it is completely forward-looking in that the equilibrium conditions at time t are completely independent of what has taken place in the past, so long as the central bank and the private sector do not themselves condition their behavior on an arbitrary past value of a variable—a case which we now explicitly rule out by restricting attention to Markov Perfect Equilibria (MPE).

3.4 Markov Perfect Equilibrium in the Game Γ_1

It follows that any MPE of the game between the central bank and households and firms must, by definition, involve strategies that are completely independent of the past realizations of all variables. This implies that in any MPE the expectations are constants so that

$$h_{1t} = h_1 \quad (31)$$

$$h_{2t} = h_2 \quad (32)$$

$$h_{3t} = h_3 \quad (33)$$

This restriction on expectation is important because it rules out that expectations can depend on “arbitrary history” or sunspots. We can now state a formal definition for a MPE

Definition 2 *A Markov Perfect Equilibrium (MPE) for $\{L_t, x_t, Y_t, h_{1t}, h_{2t}, h_{3t}, r_t\}$ at time t_0 is a collection of stochastic processes for these variables such that, for all $t \geq t_0$: (i) (L_t, x_t, Y_t) satisfy households’ and firms’ optimality conditions (25)–(27), taking r_t and (h_{1t}, h_{2t}, h_{3t}) in (31)–(33) as given; (ii) (h_{1t}, h_{2t}, h_{3t}) satisfy conditions (28)–(30) for rational expectations; (iii) households’ and firms’ strategies are Markov, i.e., independent of history, independent of time, and independent of public sunspot variables, (iv) r_t maximizes (23) subject to (25)–(27), taking expectations (h_{1t}, h_{2t}, h_{3t}) in (31)–(33) as given, and (v) the central bank’s strategy is Markov.*

It is easily verified that this definition corresponds to the more general definition of Markov perfect equilibrium for arbitrary models (e.g., Fudenberg and Tirole, 1993) applied to the special case of our two-period Taylor-contracting New Keynesian model. The key question is whether this definition implies a unique equilibrium or if there are many value equilibria that satisfy this definition.

We further state a proposition that follows directly from our definition of Markov Perfect equilibria. This proposition shows that, in a Markov perfect equilibrium, the expectations h_{1t} , h_{2t} , and h_{3t} must be constant over time:

Proposition 5 *In a Markov Perfect Equilibrium, there exist positive real numbers h_1 , h_2 , and h_3 such that $(h_{1t}, h_{2t}, h_{3t}) = (h_1, h_2, h_3)$ for all times t .*

The proof of this proposition follows from the observations that: (1) h_{1t} , h_{2t} , and h_{3t} are mathematical expectations of variables in period $t + 1$, (2) variables in period $t + 1$ depend only on variables dated $t + 1$ or later, by Proposition 1 and Definition 2, and (3) the central bank's choice for r_t and private sector's choice of (L_t, x_t, Y_t) cannot be functions of time, as part of the definition of an MPE. Thus, the mathematical expectation is the same in every period t .

Intuitively, no matter what interest rate r_t the monetary authority plays in period t , the perfect forward-looking nature of the model and the assumption of an MPE implies that the equilibria in period $t + 1$ onward are unaffected.

The proposition above indicates that our definition of MPE corresponds to what some authors, such as King and Wolman, refer to as "perfect foresight discretionary equilibrium". Because the expectations are constant across time this indicates that current and future monetary policy choose the same action and that the selection rule of equilibria is that only one equilibrium will prevail in each period. Furthermore it is a common knowledge what equilibria will prevail. The central question, then, is if the definition of equilibria above allows for more than one equilibrium. If there are more than one equilibria that satisfy the definition above there would be reasons to go further and try to specify which equilibria would be chosen at each point in time (e.g. through the realization of exogenous sunspot variables).

3.5 Lagrangean Formulation of the Optimal Policy Problem in the Game Γ_1

We solve for the optimality conditions of the central bank using a Lagrangean formulation of the optimal monetary policy problem. Based on Propositions 1 and 2 and the restriction to MPE (Definition 2), we may write the optimal policy problem as:

$$\max E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\frac{Y_T^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_T^{1+\chi}}{1+\chi} \right], \quad (34)$$

subject to the three constraints:

$$\frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}} \quad (35)$$

$$(1 - \tau)2^{-\theta}(1 + x_t^{1/\theta})^\theta [Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_{2t}] = (1 + \theta)\chi_0 [Y_t L_t^\chi + \beta(1 + x_t^{1/\theta})^{1+\theta} h_{1t}] \quad (36)$$

$$Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_{3t} \quad (37)$$

Policymakers regard h_{1t} , h_{2t} , and h_{3t} as exogenous constants, and thus they will drop out of the policymakers' first-order conditions. Nevertheless, the expectations h_{1t} , h_{2t} , and h_{3t} do influence the optimal choice of r_t , because they affect the date t realizations of L_t , x_t , and Y_t through the constraints (35)–(37).

Of course, in equilibrium, the expectations h_t must satisfy:

$$h_{1t} = E_t \frac{Y_{t+1} L_{t+1}^\chi}{(1 + x_{t+1}^{-1/\theta})^{1+\theta}} \quad (38)$$

$$h_{2t} = E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}} \quad (39)$$

$$h_{3t} = E_t Y_{t+1}^{-\varphi} (1 + x_{t+1}^{-1/\theta}) \quad (40)$$

and since we are ultimately interested in the (possibly multiple) equilibria of the model, rather than the out-of-equilibrium behavior, we will substitute these restrictions back into policymakers' first-order conditions after differentiating. Doing so yields the following first-order conditions for the optimal policy:

$$\lambda_t^{\text{Euler}} = 0 \quad (41)$$

$$\chi_0 L_t^{1+\chi} = \lambda_t^Y \frac{L_t}{Y_t} - \lambda_t^x (1 + \theta)\chi_0 Y_t \chi L_t^\chi \quad (42)$$

$$\lambda_t^Y \frac{L_t}{Y_t} = Y_t^{1-\varphi} + \lambda_t^x [(1 - \tau)(1 - \varphi)2^{-\theta}(1 + x_t^{1/\theta})^\theta Y_t^{1-\varphi} - (1 + \theta)\chi_0 Y_t L_t^\chi] \quad (43)$$

$$\lambda_t^Y 2^\theta \frac{1 + \theta}{\theta} \frac{x_t - 1}{(1 + x_t^{1/\theta})^{2(1+\theta)}} = \lambda_t^x \left\{ (1 - \tau)2^{-\theta} \left[\frac{Y_t^{1-\varphi}}{1 + x_t^{1/\theta}} + \beta \frac{1 + \theta}{\theta} h_{2t} \right] - \chi_0 \beta \frac{(1 + \theta)^2}{\theta} h_{1t} \right\} \quad (44)$$

where λ_t^Y , λ_t^x , and λ_t^{Euler} denote the Lagrange multipliers on equations (35), (36), and (37), respectively.

4 Analytic Solution for Markov Perfect Equilibria in the Game Γ_1

The restriction of attention to Markov perfect equilibria and the absence of state variables in the model combine to make a closed-form solution for the optimal policy and associated equilibria feasible. Since interest often centers around the economy's inflation rate $\pi_t \equiv P_t/P_{t-1}$, we will also define that auxiliary variable here. Note that:

$$\pi_t^{1/\theta} = \frac{1 + x_{t-1}^{1/\theta}}{1 + x_t^{-1/\theta}} \quad (45)$$

Now, by Proposition 2 and the fact that the interest r_t is also nonstochastic in equilibrium, we know that the model is in steady state in every period t . The model can then be reduced down to a single equation for π :

$$\frac{1 + \beta\pi^{(1+\theta)/\theta}}{1 + \beta\pi^{1/\theta}} \frac{1 + \pi^{1/\theta}}{1 + \pi^{(1+\theta)/\theta}} \left\{ 1 - \frac{(\pi - 1) \left[1 + \chi - (1 - \varphi) \frac{1 + \beta\pi^{(1+\theta)/\theta}}{1 + \beta\pi^{1/\theta}} \right]}{(\pi - 1) \left[1 - (1 - \varphi) \frac{1 + \beta\pi^{(1+\theta)/\theta}}{1 + \beta\pi^{1/\theta}} \right] + (1 + \pi^{(1+\theta)/\theta}) \left[1 - \frac{1}{1 + \theta} \frac{1 + \beta\pi^{(1+\theta)/\theta}}{1 + \beta\pi^{1/\theta}} \right]} \right\} = \frac{1}{1 + \theta} \quad (46)$$

The interesting question is, for what sets of parameter values β , θ , φ , and χ , does there exist more than one possible equilibrium value of π

To simplify notation, let:

$$\begin{aligned} \beta^{\text{rat}} &\equiv \frac{1 + \beta\pi^{(1+\theta)/\theta}}{1 + \beta\pi^{1/\theta}} \\ \pi^{\text{rat}} &\equiv \frac{1 + \pi^{1/\theta}}{1 + \pi^{(1+\theta)/\theta}} \\ N &\equiv (\pi - 1) \left[1 + \chi - (1 - \varphi) \beta^{\text{rat}} \right] \\ D &\equiv (\pi - 1) \left[1 - (1 - \varphi) \beta^{\text{rat}} \right] + (1 + \pi^{(1+\theta)/\theta}) \left[1 - \frac{1}{1 + \theta} \beta^{\text{rat}} \right] \end{aligned}$$

Note that the above are all functions of π . The large expression in (46) can then be written:

$$(1 + \theta) \beta^{\text{rat}} \pi^{\text{rat}} \left\{ 1 - \frac{N}{D} \right\} = 1 \quad (47)$$

We have the following propositions (proofs provided in the Appendix):

The function D defined above takes on the value zero for precisely one value of π in the interval $(1, \infty)$. D has no zeros in $(0, 1)$ if $\varphi \leq \theta/(1 + \theta)$, and D has precisely one zero in $(0, 1)$ if $\varphi > \theta/(1 + \theta)$.

Proposition 6 *Suppose that D has a zero at $\pi_0 \in (0, 1)$. Then there is no equilibrium steady-state π with $\pi \leq \pi_0$.*

The following proposition proves there is a unique equilibrium relationship between π and τ over the interval (π_0, π_1) that lies between the zeros of D .

Proposition 7 *Let $\pi_1 \in (1, \infty)$ be a zero of D . If D has a zero in $(0, 1)$, denote it by π_0 ; if D has no zero in $(0, 1)$, let $\pi_0 \equiv 0$. Then the function $\tau(\pi)$ given by (46) is strictly increasing over the interval (π_0, π_1) .*

Together with Proposition 4, Proposition 5 shows that, if there exist multiple equilibria, the additional equilibria must lie beyond the value π_1 defined in Proposition 5. The following Proposition shows that, for values of φ that lie quite close to 1, there is a unique equilibrium π for any given $\tau \in [-\theta, 1]$.

Proposition 8 *Let $\pi_1 \in (1, \infty)$ be a zero of D . Then $\tau(\pi) < -\theta$ for all $\pi > \pi_1$ if and only if $\varphi \geq 1 + (1 - \beta)/(\beta(1 + \theta))$*

5 Discussion of Stackelberg vs. Simultaneous Play

In our previous discussion we assumed that the nominal interest rate is the instrument of policy. It will not change the result if we assume instead, that the money supply is the instrument of policy. To see this suppose that there is a cash advance constraint, as in King and Wolman (2005), so that

$$\frac{M_t}{P_t} = Y_t$$

in each period. We can rewrite this in terms of the variables of the previous section as

$$m_t x_t^{-1} 2^{-\theta} (1 + x_t^{1/\theta})^\theta = Y_t \quad (48)$$

where we have defined $m_t \equiv \frac{M_t}{p_{t-1}^*}$. Consider now a policy in which m_t is the policy instrument instead of r_t . In this case the Lagrangian formulation in section 3.4. is exactly the same except we now take equation (48) into account. Denoting the Lagrangian multiplier of this constraint by λ_t^{Money} we obtain once again the same conditions as before because the first order condition with respect to m_t indicates that

$$\lambda_t^{Money} = 0.$$

Hence introducing money explicitly into the analysis under our timing assumption has no effect on the results derived in the last few sections. Consider now the case in which the central bank selects the value of the instrument before the market clear, so that it cannot react to private sector action in period t . The solution is much more complicated in this case and can only be solved numerically. Some progress can be made analytically, however. To simplify we assume that $\chi = 0$ and $\varphi = 1$ and $\tau = 0$. Furthermore we assume that the money supply is the policy instrument, and this assumption is important when the government selects the policy before the market clear. To see how the Stackelberg timing assumption gives rise to multiple equilibria it is easier to assume that there is a termination period T and solve backwards. Consider first period T . The equilibrium condition at time T are

$$L_T = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}} Y_T$$

$$2^{-\theta} (1 + x_T^{1/\theta})^\theta = \frac{1 + \theta}{1 - \tau} \chi_0 Y_T$$

$$m_T x_T^{-1} 2^{-\theta} (1 + x_T^{1/\theta})^\theta = Y_T$$

For a given m_T the solution is

$$2^{-\theta}(1+x_T^{\frac{1}{\theta}})^{\theta} = (1+\theta)\chi_0 m_T x_T^{-1} 2^{-\theta}(1+x_T^{1/\theta})^{\theta}$$

$$x_T = (1+\theta)\chi_0 m_T$$

which gives a value for x_T for a given m_T . The optimal value for x_T and m_T is given by maximizing the objective

$$\ln Y_T - \chi_0 L_T$$

which if we substitute the constrains gives

$$\ln(2^{-\theta}(1+x_T^{\frac{1}{\theta}})^{\theta} \frac{1}{1+\theta} \chi_0^{-1}) - \frac{1+x_T^{(1+\theta)/\theta}}{1+x_T^{1/\theta}} \frac{1-\tau}{1+\theta}$$

This gives rise to the first order condition

$$F(x_T, \theta) = \theta(1+x_T^{1/\theta}) - x_T(1+x_T^{1/\theta}) + \frac{1}{1+\theta}(1+x_T^{(1+\theta)/\theta}) = 0$$

We now observe that for $x_T > 1$ there is uniqueness since $x_T^{1/\theta} > x_T^{1/\theta-1}$. Denote x_T that solves this equation \bar{x} and $Y_T = \bar{Y} = 2^{-\theta}(1+\bar{x}^{\frac{1}{\theta}})^{\theta} \frac{1}{1+\theta} \chi_0^{-1}$ and $L_T = \bar{L} = \frac{1+\bar{x}^{(1+\theta)/\theta}}{1+\bar{x}^{1/\theta}} \frac{1}{1+\theta} \chi_0^{-1}$. We now solve the maximization problem in period $T-1$ and show that there is multiple equilibria. The equilibrium condition at time $T-1$ are:

$$L_{T-1} = 2^{\theta} \frac{1+x_{T-1}^{(1+\theta)/\theta}}{(1+x_{T-1}^{1/\theta})^{1+\theta}} Y_{T-1}$$

$$2^{-\theta}(1+x_{T-1}^{\frac{1}{\theta}})^{\theta} [1 + \beta(1+x_{T-1}^{1/\theta}) \frac{1}{1+\bar{x}^{-1/\theta}}] = (1+\theta)\chi_0 [Y_{T-1} + \beta(1+x_{T-1}^{1/\theta})^{1+\theta} \frac{\bar{Y}}{(1+\bar{x}^{-1/\theta})^{1+\theta}}]$$

$$m_{T-1} x_{T-1}^{-1} 2^{-\theta}(1+x_{T-1}^{1/\theta})^{\theta} = Y_{T-1}$$

For a given m_{T-1} the solution is determined by combining the last two equations to yield

$$m_{T-1} = x_{T-1} \frac{1}{1+\theta} \chi_0^{-1} [1 + \beta(1+x_{T-1}^{1/\theta}) h_2] - x_{T-1} 2^{\theta} \beta (1+x_{T-1}^{1/\theta})^{1-\theta} h_1$$

Figure X shows plots the right hand side of this equation. This line corresponds to all private sector allocations that are consistent with equilibrium, we call it the PE curve. The horizontal line corresponds to a particular choice for m_{T-1} , we call it the PC curve. Note that because of the shape of the PE curve there are two equilibria for any given choice of m_{T-1}

6 Conclusions

To be added.

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